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Canonical Higgs fields from higher-dimensional gravity

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Abstract

We consider the dimensional reduction of a gravitational field g in a multidimensional universe endowed with a simple action of a compact Lie group. It is known that when the group preserves g, this dimensional reduction leads in particular to scalar fields that correspond to an invariant metric on each orbit. We show that the action functionals of those fields (obtained from the reduction of Einstein's action) exhibit, in the hyperbolic case, polynomials of several variables having a degree less or equal to 6.

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1. Introduction

In the usual gauge theories, Higgs fields appear at the classical level as scalar fields ϕ, whose dynamics are prescribed by a fourth-degree polynomial potential

$$V(\phi) = cte(|\phi|^2 - a^2)^2$$

at least when we are in front of a spontaneous symmetry breaking mechanism that will generate mass for the gauge particles.

At first sight, such Higgs potentials seem to be put in by hand, while it would be interesting to have a geometrical understanding of their origin. The approach considered here is that of dimensional reduction theories, behind which the central idea is the existence of a simple and symmetric field theory in a higher-dimensional universe, leading through a process of dimensional reduction, to the field theories we see in our four-dimensional space–time.

A first step in this direction is to search for multidimensional theories that would give, by dimensional reduction, field theories containing Higgs-like scalar fields.

A well-known solution for this problem is to take a symmetric Yang–Mills theory on a multidimensional universe. Under appropriate conditions, the dimensional reduction of such a theory, studied first by Manton (cf. [1,2]), gives a Yang–Mills theory with Higgs fields exhibiting a symmetry-breaking potential like the one above.

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Actually, studies made by Jadczyk, Coquereaux and Pilch (cf. [3–5]) showed the following: if one wants to perform the dimensional reduction of a symmetric Yang–Mills field in a sufficiently general setting, then the natural object to reduce is not a symmetric Yang–Mills field alone, but an appropriate combination of two fields: a symmetric Yang–Mills field + a symmetric gravity field.

They also showed that the best way to get such an appropriate combination is to consider that the multidimensional universe in which these two symmetric fields live is itself the result of the dimensional reduction of another universe with even more dimensions, and by adding a constant field to the symmetric couple, we can get all the fields from a single and very symmetric gravitational field on the universe with the highest dimension. In this sector, we are in front of Kaluza–Klein theories, as they have been extended by Richard Kerner (cf. [6,7]) to the case where the compactified extra dimensions are those of a non-necessarily abelian Lie group.

Thus, the four-dimensional theory containing Higgs-like scalar fields seems to come in fact from the dimensional reduction of a symmetric *gravity* field. But those ϕ fields which exhibit a fourth-degree polynomial potential are not the only fields one obtains from the reduction of the very symmetric gravitational theory above. There are other fields also, in particular scalar fields with different dynamics.

In fact, starting from the above particular "big" universe, with the particular very symmetric gravitational field, we have the choice between two ways of reducing (cf. [5]): either we make a first partial reduction to get the less symmetric couple, and then reduce this last one to get our four-dimensional theory with Higgs fields, or we make directly a complete dimensional reduction of the symmetric gravity field in the big universe, which gives in particular Higgs fields on space–time, that we'll have to distinguish from other scalar fields obtained.

These two methods seem equivalent, which implies that the intermediate level could be omitted! In other words, if one is seeking for a more fundamental and geometrical understanding of the nature of the Higgs fields, this is to be found in the dimensional reduction of a gravitational theory.

This idea is supported also by the work of Maspfuhl, who has showed in the first part of [8] that in the presence of a coistropic constraint, one can get the phase space of particles in gravitational, Yang–Mills and Higgs fields, by starting from a certain class of Hamiltonians defined on a general Poisson manifold.

In Section 2, we will give a short recap on dimensional reduction of symmetric gravity, with a focus on the scalar fields obtained from such a reduction that are interpreted as collections of invariant metrics on homogeneous spaces, and are known to have a rather different behavior than the usual Higgs fields. Therefore, we will propose in Section 3 a definition of Higgs-like scalar fields, by distinguishing components of the general scalar fields on which we recover a positive polynomial potential. We study then the Yang–Mills term and the kinetic term of the reduced theory. Finally, we study in Section 4 a model with $SU(5)/U(1)$ as compactified space, leading to an abelian gauge theory having a torus as gauge group.

2. Dimensional reduction of symmetric gravitational fields

2.1. The dimensional reduction theorem

Coquereaux and Jadzcyk performed the dimensional reduction of symmetric gravitational fields in the general context of fiber bundles with homogeneous fibers (cf. [9]), and the short recap we present in this first part is directly inspired from their work.

One starts with a manifold U, on which a compact Lie group G is acting on the right. Let M the quotient manifold of U by the action of G. One chooses then some point u_0 in U, and let H be the stabilizer of u_0 under G. The action of G is supposed to be *regular* (or *simple*) in the following sense: every stabilizer under G is conjugated to H. By the orbit theorem, which states the elementary fact that each orbit uG is diffeomorphic to $G/\text{stab}(u)$, we deduce that all the orbits are in fact diffeomorphic to the same homogeneous space which is G/H. Therefore, U can be realized as a fiber bundle over M, the fibers being homogeneous spaces for the group G, copies of the typical fiber G/H.

We would like to see the fiber bundle $(U, M, G/H)$ as associated to some principal fiber bundle. For this, we introduce the normalizer $N(H)$ of H is G. Notice that since we started with a right action of G on U, G/H is the space of right cosets Ha, and thus G/H is equipped with a right action of G. So G does not act naturally on G/H on the left. But if we take the normalizer $N(H)$, then $N(H)$ acts of course on G/H on the right, but it also acts naturally on G/H on the left! Indeed, all one has to do is to set $n(Ha) = Hna$ and then one gets a left action of $N(H)$ on G/H, and since $(nh)(Ha) = n(Ha)$, we see that $N(H)|H$ acts also on G/H, on the left. On the other hand, the right action of G on U induces a right action of $N(H)$ on U. Let Q be the submanifold of U whose points have exactly H as stabilizer. It's easy to see then that Q is invariant under the right action of $N(H)$: indeed, for $q \in Q$ and $n \in N(H)$, $\mathrm{stab}(qn) = n^{-1}\mathrm{stab}(q)n = n^{-1}Hn = H$, so $qn \in Q$. Notice also that $q(nh) = qh'n = qn$, so here also, we get an action of $N(H)|H$, but this time on Q and on the right. The difference between the right-action of $N(H)$ on Q, and that of $N(H)|H$, is that the latter is free. Therefore, Q is realized as a principal fiber bundle on M, with $N(H)|H$ as structure group. Now, using the left action of $N(H)|H$ on G/H, we may construct the associated fiber bundle $Q \times_{N(H)|H} G/H$, and it's not hard to see that $(q, Ha)N(H)|H \longmapsto qa$ is an isomorphism from $Q \times_{N(H)|H} G/H$ to U.

Now we get to the infinitesimal level. Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{h}$ that of H. The group G being compact, it is possible to choose a scalar product $\langle , \rangle$ on $\mathfrak{g}$ that is invariant by the adjoint representation of G. Let us call $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Since H acts on $\mathfrak{h}$ and the scalar product is in particular $\mathrm{Ad}(H)$-invariant, it is clear that we get then a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ (reductive simply means that the subspace $\mathfrak{m}$ is invariant under the restriction to H of the adjoint representation). Thus, we have a representation $\mathrm{Ad}^{\mathfrak{m}}_{|H} : H \longrightarrow GL(\mathfrak{m})$ that will be important in the following. We also introduce a vector space that is going to play a significant role:

Let $S_2^H(\mathfrak{m})$ denote the vector space of all $\mathrm{Ad}(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In other terms, $S_2^H(\mathfrak{m}) = \{f \in \mathrm{Sym}^2(\mathfrak{m}^*)/\, {}^t\mathrm{Ad}_h(f) = f \ \forall h \in H\}$.

We shall endow this vector space with a representation of the group $N(H)|H$: first, notice that $\mathfrak{m}$, being the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ for our $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$, is not only $\mathrm{Ad}(H)$-invariant, but also $\mathrm{Ad}(N(H))$-invariant (since $N(H)$ acts on $\mathfrak{h}$ and the scalar product is in particular $\mathrm{Ad}(N(H))$-invariant). Thus, $N(H)$ acts on $\mathfrak{m}$, and therefore, it acts on the space of symmetric bilinear form on $\mathfrak{m}$. Actually, it's easy to check that the subspace $S_2^H(\mathfrak{m})$ is invariant under $N(H)$, and that this representation induces a representation ρ of $N(H)|H$ on $S_2^H(\mathfrak{m})$, given by $\rho_{[n]}(f) = {}^t\mathrm{Ad}_{n^{-1}}(f)$.

Now, starting from a G-invariant metric g on U, we can construct a triple of fields (γ, α, f), that can be considered locally as defined on M.

. At each point u in U, $g(u)$ is a scalar product on the tangent space T_uU. If we call Z_u the orthogonal complement of the vertical subspace $V_u(U)$ at u, we get a horizontal distribution on the fiber bundle $(U, M, G/H)$, that is, a connection α on the principal fiber bundle $(Q, M, N(H)|H)$.
. Next, the connection α provides an isomorphism at each point u between the horizontal space Z_u and the tangent space T_xM of M at $x = \pi(u)$ ($\pi : U \longrightarrow M$ being the bundle's projection). This isomorphism allows us to project the scalar product $g(u)_{|Z_u \times Z_u}$ to a scalar product $\gamma(x)$ on T_xM. Hence, we get a metric γ on M.
. Finally, for $x \in M$, let f_x denote the pull-back of g by the canonical injection of the fiber U_x in U. Then, for each $x \in M$, f_x is a G-invariant metric on the fiber U_x. Therefore, for each $q \in Q_x$, it defines on T_qU_x a scalar product $f_x(q)$ which is invariant by the isotropy representation of H. If we pull-back $f_x(q)$ by the H-equivariant isomorphism from $\mathfrak{m}$ to T_qU_x defined by the choice of q, we get an $\mathrm{Ad}(H)$-invariant scalar product $f(q)$ on $\mathfrak{m}$. Hence, we have an $N(H)|H$-equivariant map $f : Q \longrightarrow S_2^H(\mathfrak{m})$, that we shall call a *Thiry scalar field*.

Thus, we have the following dimensional reduction theorem, proven in [9]:

Theorem 2.1 (*Coquereaux and Jadczyk*). *The preceding construction defines a one-to-one correspondence between the set of G-invariant metrics on U, and the set of triples (γ, α, f), where:*

- *γ is a metric on the base manifold M,*
- *α is a connection form on the principal fiber bundle $(Q, M, N(H)|H)$,*
- *f is a scalar field on Q taking values in $S_2^H(\mathfrak{m})$ (equivariant for the group $N(H)|H$).*

2.2. The potential for the Thiry scalar field

It is possible to perform now the dimensional reduction of the action density. On the multidimensional universe, we have a pure gravity theory; therefore we can write the Einstein–Hilbert action. The constraint of G-invariance of the metric field implies that the Lagrangian of the multidimensional theory does not depend on the internal variables, but only the space–time variables. Therefore, one can integrate over the internal space to get the reduced action density on the base manifold. What we obtain is of course an Einstein–Yang–Mills theory interacting with scalar field

f. In general, we have a kinetic term for the Thiry scalar field f, when it interacts with the gauge field. We also get a potential term, that we are going to study carefully here, leaving the Yang–Mills term and the kinetic term to Section 3.4.

The potential appearing in the reduced action is a real-valued function V defined on the open set $S_2^{H++}(\mathfrak{m})$ made of positive definite elements of $S_2^H(\mathfrak{m})$. In fact, V can be more generally defined on the open subset of all non-degenerate elements of $S_2^H(\mathfrak{m})$, but we shall restrict ourselves to the positive definite case.

It is well known (cf. [10] for example) that $S_2^{H++}(\mathfrak{m})$ is in one-to-one correspondence with the set of all G-invariant metrics on the homogeneous space G/H.

As one might expect, the potential obtained by dimensional reduction is then the opposite of the scalar curvature functional of G/H restricted to the set of G-invariant metrics.

There are several ways to compute the scalar curvature of a riemannian homogeneous space (cf. for example [10–13]) but most of them give the expression after having chosen a fixed G-invariant metric and an orthonormal basis on the tangent space at the origin. Here, we are rather interested in the scalar curvature functional, so we would like the dependence on the metric to be explicit. The formula we present here is an intrinsic version of that of [9].

Let us introduce some notations first. We denote by $\widetilde{\mathrm{ad}} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$. For any $X \in \mathfrak{m}$, let $\mathrm{ad}(X)$ be the endomorphism of $\mathfrak{m}$ defined by: $\mathrm{ad}(X) = \pi_{\mathfrak{m}} \circ \widetilde{\mathrm{ad}}(X) \circ \iota_{\mathfrak{m}}$, where $\iota_{\mathfrak{m}} : \mathfrak{m} \longrightarrow \mathfrak{g}$ and $\pi_{\mathfrak{m}} : \mathfrak{g} \longrightarrow \mathfrak{m}$ are the canonical injection and projection respectively. Now for any bilinear form b on $\mathfrak{m}$, we denote by $\hat{b} : \mathfrak{m} \longrightarrow \mathfrak{m}^*$ the linear map canonically associated to b (that is: $\hat{b}(X)(Y) = b(X, Y)$ for any $X, Y \in \mathfrak{m}$). With these notations, the potential $V : S_2^{H++}(\mathfrak{m}) \longrightarrow \mathbb{R}$ is given by:

$$V(f) = \frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D}) + \frac{1}{2}\mathrm{tr}(\hat{f}^{-1} \circ \hat{B}) + \mathrm{tr}(\hat{f}^{-1} \circ \hat{D}^{\mathfrak{h}})$$

where D, B and $D^{\mathfrak{h}}$ are the bilinear forms on $\mathfrak{m}$ defined by:

$$\begin{aligned} D(X,Y) &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t\mathrm{ad}(X) \circ \hat{f} \circ \mathrm{ad}(Y)) \\ B(X,Y) &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \\ D^{\mathfrak{h}}(X,Y) &= \mathrm{tr}(\pi_{\mathfrak{m}} \circ \widetilde{\mathrm{ad}}(Y) \circ \iota_{\mathfrak{h}} \circ \pi_{\mathfrak{h}} \circ \widetilde{\mathrm{ad}}(X) \circ \iota_{\mathfrak{m}}). \end{aligned}$$

The first term of V will be of interest to us, so we will begin by expressing it in a different way. Let $\mathcal{L}(\mathfrak{m})$ denote the space of endomorphisms of $\mathfrak{m}$. Each $f \in S_2^{H++}(\mathfrak{m})$ defines a scalar product on $\mathcal{L}(\mathfrak{m})$ given by: $\langle u, v\rangle_{ff} = \mathrm{tr}(\hat{f}^{-1} \circ {}^t u \circ \hat{f} \circ v)$. If $\widehat{ff} : \mathcal{L}(\mathfrak{m}) \longrightarrow \mathcal{L}(\mathfrak{m})^*$ denotes the isomorphism associated to $\langle , \rangle_{ff}$, then we get a scalar product on the space $\mathcal{L}(\mathfrak{m}, \mathcal{L}(\mathfrak{m}))$ of linear maps from $\mathfrak{m}$ to $\mathcal{L}(\mathfrak{m})$ by setting: $\langle \alpha, \beta\rangle_{fff} = \mathrm{tr}(\hat{f}^{-1} \circ {}^t\alpha \circ \widehat{ff} \circ \beta)$. (That $\langle , \rangle_{ff}$ and $\langle , \rangle_{fff}$ are positive definite symmetric bilinear forms is not difficult to prove.) Using the preceding definitions, we can write:

$$\begin{aligned} D(X,Y) &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t\mathrm{ad}(X) \circ \hat{f} \circ \mathrm{ad}(Y)) \\ &= \langle \mathrm{ad}(X), \mathrm{ad}(Y)\rangle_{ff} \\ &= {}^t\mathrm{ad}(\langle , \rangle_{ff})(X,Y) \end{aligned}$$

so $\hat{D} = {}^t\mathrm{ad} \circ \widehat{ff} \circ \mathrm{ad}$, therefore

$$\begin{aligned} \mathrm{tr}(\hat{f}^{-1} \circ \hat{D}) &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t\mathrm{ad} \circ \widehat{ff} \circ \mathrm{ad}) \\ &= \langle \mathrm{ad}, \mathrm{ad}\rangle_{fff}. \end{aligned}$$

Thus, we have

$$\frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D}) = \frac{1}{4}\|\mathrm{ad}\|^2_{fff}$$

which proves the positivity of this term.

It is convenient to write the representation ρ of the gauge group $N(H)|H$ on the vector space $S_2^H(\mathfrak{m})$ in terms of the linear maps $\hat{f} : \mathfrak{m} \longrightarrow \mathfrak{m}^*$: $\rho_{[n]}(\hat{f}) = {}^t\mathrm{Ad}_{n^{-1}} \circ \hat{f} \circ \mathrm{Ad}_{n^{-1}}$.

When we will study the kinetic term, we will need the expression of the covariant derivative corresponding to the connection α and acting on the $S_2^H(\mathfrak{m})$-valued Thiry scalar fields. Therefore, we need to write also the infinitesimal version of ρ, that is, the representation ρ' of the Lie algebra $\mathfrak{k} = \mathfrak{n}(\mathfrak{h})|\mathfrak{h}$ on $S_2^H(\mathfrak{m})$. It is easy to check that this last one is given by: $\rho'_{[\tilde{A}]}(\hat{f}) = -({}^t\mathrm{ad}_{\tilde{A}} \circ \hat{f}) - (\hat{f} \circ \mathrm{ad}_{\tilde{A}})$ for every $\tilde{A} \in \mathfrak{n}(\mathfrak{h})$.

Proposition 2.2. *The potential V is invariant under the representation of the gauge group $N(H)|H$.*

Proof. We begin with the first term of V: $\mathrm{tr}(\hat{f}^{-1} \circ \hat{D}_f)$

$$\begin{aligned} D_{\rho_{[n]}(\hat{f})}(X, Y) &= \mathrm{tr}(\rho_{[n]}(\hat{f})^{-1} \circ {}^t\mathrm{ad}(X) \circ \rho_{[n]}(\hat{f}) \circ \mathrm{ad}(Y)) \\ &= \mathrm{tr}(({}^t\mathrm{Ad}_{n^{-1}} \circ \hat{f} \circ \mathrm{Ad}_{n^{-1}})^{-1} \circ {}^t\mathrm{ad}(X) \circ {}^t\mathrm{Ad}_{n^{-1}} \circ \hat{f} \circ \mathrm{Ad}_{n^{-1}} \circ \mathrm{ad}(Y)) \\ &= \mathrm{tr}(\mathrm{Ad}_n \circ \hat{f}^{-1} \circ {}^t\mathrm{Ad}_n \circ {}^t\mathrm{ad}(X) \circ {}^t\mathrm{Ad}_{n^{-1}} \circ \hat{f} \circ \mathrm{Ad}_{n^{-1}} \circ \mathrm{ad}(Y)) \\ &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t(\mathrm{Ad}_{n^{-1}} \circ \mathrm{ad}(X) \circ \mathrm{Ad}_n) \circ \hat{f} \circ (\mathrm{Ad}_{n^{-1}} \circ \mathrm{ad}(Y) \circ \mathrm{Ad}_n)) \\ &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t\alpha_n(X) \circ \hat{f} \circ \alpha_n(Y)) \end{aligned}$$

if we set $\alpha_n(X) = \mathrm{Ad}_{n^{-1}} \circ \mathrm{ad}(X) \circ \mathrm{Ad}_n$ for all $X \in \mathfrak{m}$.

So $\hat{D}_{\rho_{[n]}(\hat{f})} = {}^t\alpha_n \circ \widehat{ff} \circ \alpha_n$. Therefore,

$$\begin{aligned} \mathrm{tr}(\rho_{[n]}(\hat{f})^{-1} \circ \hat{D}_{\rho_{[n]}(\hat{f})}) &= \mathrm{tr}(\mathrm{Ad}_n \circ \hat{f}^{-1} \circ {}^t\mathrm{Ad}_n \circ {}^t\alpha_n \circ \widehat{ff} \circ \alpha_n) \\ &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t(\alpha_n \circ \mathrm{Ad}_n) \circ \widehat{ff} \circ (\alpha_n \circ \mathrm{Ad}_n)). \end{aligned}$$

But for all $X, Y \in \mathfrak{m}$,

$$\begin{aligned} \alpha_n \circ \mathrm{Ad}_n(X)(Y) &= \alpha_n(\mathrm{Ad}_n(X))(Y) \\ &= \mathrm{Ad}_{n^{-1}} \circ \mathrm{ad}(\mathrm{Ad}_n(X)) \circ \mathrm{Ad}_n(Y) \\ &= \mathrm{Ad}_{n^{-1}}(\mathrm{ad}(\mathrm{Ad}_n(X))(\mathrm{Ad}_n(Y))) \\ &= \mathrm{Ad}_{n^{-1}}([\mathrm{Ad}_n(X), \mathrm{Ad}_n(Y)]_{\mathfrak{m}}) \\ &= \mathrm{Ad}_{n^{-1}} \circ \mathrm{Ad}_n([X, Y]_{\mathfrak{m}}) \\ &= [X, Y]_{\mathfrak{m}} \\ &= \mathrm{ad}(X)(Y). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{tr}(\rho_{[n]}(\hat{f})^{-1} \circ \hat{D}_{\rho_{[n]}(\hat{f})}) &= \mathrm{tr}(\hat{f}^{-1} \circ {}^t\mathrm{ad} \circ \widehat{ff} \circ \mathrm{ad}) \\ &= \mathrm{tr}(\hat{f}^{-1} \circ \hat{D}_f). \end{aligned}$$

The invariance of the two other terms follows from the invariance of the Killing form. □

We saw that V is, up to a sign, the scalar curvature functional of a homogeneous space. This function has no reason to have a fixed sign. There are many examples (cf. [13,9]) in which this potential is not bounded from below. At this level, one may wonder if it is possible to construct a physical theory out of such a potential.

In order to recover a more realistic pattern for the potential, we shall try to distinguish certain directions in the space $S_2^H(\mathfrak{m})$ of the scalar fields, on which we might have a positive polynomial potential, closer to that of the Higgs fields.

3. Recovering scalar fields with polynomial potential

3.1. Introduction

Let us consider the vector space $S_2^H(\mathfrak{m})$. There are at least two interesting ways to decompose it. First, we may start by finding the decomposition of $\mathfrak{m}$ into irreducible representations of the group H:

$$\mathfrak{m} = \bigoplus \mathfrak{m}_i$$

with $\mathrm{Ad}(H)\mathfrak{m}_i \subset \mathfrak{m}_i$, and the $\mathfrak{m}_i$'s are irreducible.

Associated to this decomposition, one then writes a decomposition of $S_2^H(\mathfrak{m})$, which allows for example the calculation of the dimension of $S_2^H(\mathfrak{m})$.

Another natural decomposition of $S_2^H(\mathfrak{m})$ is the one in terms of irreducible representations of $N(H)|H$. It would seem interesting to look at some $N(H)|H$-invariant subspace of $S_2^H(\mathfrak{m})$, constructed out of $\mathrm{Ad}(H)$-invariant factors in the decomposition $\mathfrak{m} = \oplus \mathfrak{m}_i$. In particular models of symmetric gauge fields dimensional reduction, we find Higgs fields as intertwining operators between two representative spaces, so this suggests we look at fields of the following type: $\mathrm{Ad}(H)$-equivariant maps $\phi : \mathfrak{m}_i \longrightarrow \mathfrak{m}_j$ coming from the fields $f \in S_2^H(\mathfrak{m})$.

3.2. Decomposition of $S_2^H(\mathfrak{m})$

Let us study the following situation: we fix two $\mathrm{Ad}(N(H))$-invariant vector subspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ in $\mathfrak{m}$, getting what we shall call an $\mathrm{Ad}(N(H))$-*invariant splitting*: $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We denote by $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ the space of $\mathrm{Ad}(H)$-equivariant linear maps from $\mathfrak{m}_1$ to $\mathfrak{m}_2$. Then, we can state the lemma on which all our following results will lie:

Lemma 3.1. *There is a one-to-one correspondence between the space $S_2^H(\mathfrak{m})$ and the direct product $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) \times S_2^H(\mathfrak{m}_1) \times S_2^H(\mathfrak{m}_2)$.*

It is given in matrix form by:

$$(\phi, h, k) \longmapsto f = \begin{pmatrix} h + {}^t\phi k \phi & -{}^t\phi k \\ -k\phi & k \end{pmatrix}.$$

Proof. (1) Taking $f \in S_2^H(\mathfrak{m})$, we set:

. $\phi = -\mathrm{pr}^{\mathfrak{m}_2}_{|\mathfrak{m}_1}$, where $\mathrm{pr}^{\mathfrak{m}_2}$ is the orthogonal projector on $\mathfrak{m}_2$ for the Euclidean structure defined by f.

. $\forall X_1, Y_1 \in \mathfrak{m}_1, h(X_1, Y_1) = f(X_1 + \phi(X_1), Y_1 + \phi(Y_1))$.

. $k = f_{|\mathfrak{m}_2 \times \mathfrak{m}_2}$.

For all $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ ($X_1, Y_1 \in \mathfrak{m}_1, X_2, Y_2 \in \mathfrak{m}_2$), letting $X = (X_1 + \phi(X_1)) + (-\phi(X_1) + X_2)$ and $Y = (Y_1 + \phi(Y_1)) + (-\phi(Y_1) + Y_2)$, we compute $f(X, Y)$ and obtain:

$$f(X, Y) = h(X_1, Y_1) + k(\phi(X_1), \phi(Y_1)) - k(\phi(X_1), Y_2) - k(\phi(Y_1), X_2) + k(X_2, Y_2).$$

(2) Conversely, let $(\phi, h, k) \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) \times S_2^H(\mathfrak{m}_1) \times S_2^H(\mathfrak{m}_2)$. It is not difficult to see that if we define $f \in S_2^H(\mathfrak{m})$ by the preceding formula, then:

. $\phi = -\mathrm{pr}^{\mathfrak{m}_2}_{|\mathfrak{m}_1}$, where $\mathrm{pr}^{\mathfrak{m}_2}$ is the orthogonal projector on $\mathfrak{m}_2$ for the Euclidean structure defined by f.

. $\forall X_1, Y_1 \in \mathfrak{m}_1, h(X_1, Y_1) = f(X_1 + \phi(X_1), Y_1 + \phi(Y_1))$.

. $k = f_{|\mathfrak{m}_2 \times \mathfrak{m}_2}$. $\square$

Remark 3.2. The space of Higgs fields that we obtained seems to depend on the choice of the decomposition of $\mathfrak{m}$ into $\mathfrak{m}_1$ and $\mathfrak{m}_2$. In fact, we can expect to see a natural justification of this choice in the context of a supersymmetric extension of this model.

It is easy to check the following:

(1) f is positive definite if and only if h and k are positive definite.

(2) The assumption of $\mathrm{Ad}(N(H))$-invariance for $\mathfrak{m}_1$ and $\mathfrak{m}_2$ implies the existence of a representation of $N(H)|H$ on $S_2^H(\mathfrak{m}_1)$, on $S_2^H(\mathfrak{m}_2)$, and also on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. Denoting by $\mathrm{Ad}^{(1)}$ (resp. $\mathrm{Ad}^{(2)}$) the representation of $N(H)$ on $\mathfrak{m}_1$ (resp. on $\mathfrak{m}_2$), and by $\mathrm{ad}^{(1)}$ (resp. $\mathrm{ad}^{(2)}$) the representation of $\mathfrak{n}(\mathfrak{h})$ on $\mathfrak{m}_1$ (resp. on $\mathfrak{m}_2$), the representations on the fields spaces are given by:

$$\rho^1_{[n]}(\hat{h}) = {}^t\mathrm{Ad}^{(1)}_{n^{-1}} \circ \hat{h} \circ \mathrm{Ad}^{(1)}_{n^{-1}}$$
$$\rho^2_{[n]}(\hat{k}) = {}^t\mathrm{Ad}^{(2)}_{n^{-1}} \circ \hat{k} \circ \mathrm{Ad}^{(2)}_{n^{-1}}$$
$$\rho^0_{[n]}(\phi) = \mathrm{Ad}^{(2)}_{n} \circ \phi \circ \mathrm{Ad}^{(1)}_{n^{-1}}.$$

We define the corresponding representations $\eta^i = (\rho^i)'$ of the Lie algebra $\mathfrak{k} = \mathfrak{n}(\mathfrak{h})|\mathfrak{h}$: For every $\tilde{A} \in \mathfrak{n}(\mathfrak{h})$,

$$\eta^1_{[\tilde{A}]}(\hat{h}) = -({}^t\mathrm{ad}^{(1)}_{\tilde{A}} \circ \hat{h}) - (\hat{h} \circ \mathrm{ad}^{(1)}_{\tilde{A}})$$
$$\eta^2_{[\tilde{A}]}(\hat{k}) = -({}^t\mathrm{ad}^{(2)}_{\tilde{A}} \circ \hat{k}) - (\hat{k} \circ \mathrm{ad}^{(2)}_{\tilde{A}})$$
$$\eta^0_{[\tilde{A}]}(\phi) = (\mathrm{ad}^{(2)}_{\tilde{A}} \circ \phi) - (\phi \circ \mathrm{ad}^{(1)}_{\tilde{A}}).$$

3.3. The potential for the scalar field ϕ

For h and k fixed in $S_2^{H++}(\mathfrak{m}_1)$ and $S_2^{H++}(\mathfrak{m}_2)$ respectively, let us compute the potential V in terms of ϕ. Here we denote by V the real-valued function defined on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ by:

$$V(\phi) = \frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D}) + \frac{1}{2}\mathrm{tr}(\hat{f}^{-1} \circ \hat{B}) + \mathrm{tr}(\hat{f}^{-1} \circ \hat{D}^{\mathfrak{h}})$$

where f is given by:

$$\hat{f} = {}^t\pi_1 \circ \hat{h} \circ \pi_1 + {}^t(\phi \circ \pi_1 - \pi_2) \circ \hat{k} \circ (\phi \circ \pi_1 - \pi_2)$$

We will need to split $\mathrm{ad}(X)$ in terms of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, so we define:

$$\mathrm{ad}_{11}(X) = \pi_1 \circ \mathrm{ad}(X) \circ \iota_1 \in \mathcal{L}(\mathfrak{m}_1)$$
$$\mathrm{ad}_{12}(X) = \pi_1 \circ \mathrm{ad}(X) \circ \iota_2 \in \mathcal{L}(\mathfrak{m}_2, \mathfrak{m}_1)$$
$$\mathrm{ad}_{21}(X) = \pi_2 \circ \mathrm{ad}(X) \circ \iota_1 \in \mathcal{L}(\mathfrak{m}_2, \mathfrak{m}_1)$$
$$\mathrm{ad}_{22}(X) = \pi_2 \circ \mathrm{ad}(X) \circ \iota_2 \in \mathcal{L}(\mathfrak{m}_2).$$

With the notations that we will introduce below, we are going to prove the following theorem:

Theorem 3.3. 1. *The potential for the scalar field ϕ is a polynomial of degree at most 6 in ϕ, whose sixth-degree term is positive and is given by:*

$$V^{(6)}(\phi) = \frac{1}{4}\|\phi \mathrm{ad}_{12}(\phi)\phi\|^2_{hhk}.$$

Besides, $V^{(6)} = 0$ if and only if the third-degree condition is satisfied for every ϕ: $\{\phi \circ\ \mathrm{ad}_{12}(\phi(X)) \circ \phi = 0\ \forall X \in \mathfrak{m}_1\}$.

2. *If $V^{(6)} = 0$, the potential $V(\phi)$ becomes a polynomial of degree at most 4 in ϕ, whose fourth-degree term is positive and is given by:*

$$V^{(4)}(\phi) = \frac{1}{4}\|\phi \mathrm{ad}_{11}(\phi) - \mathrm{ad}_{22}(\phi)\phi + \phi \mathrm{ad}_{12}\phi\|^2_{hhk} + \frac{1}{4}\|\mathrm{ad}_{12}(\phi)\phi\|^2_{hhh}$$
$$+ \frac{1}{4}\|\phi \mathrm{ad}_{12}(\phi)\|^2_{hkk} + \frac{1}{4}\|\phi \mathrm{ad}_{12}\phi\|^2_{khk}.$$

Besides, $V^{(4)} = 0$ if and only if the quadratic conditions are satisfied for every ϕ:

$$(\mathrm{C})\begin{cases} \phi \mathrm{ad}_{11}(\phi(X)) - \mathrm{ad}_{22}(\phi(X))\phi + \phi \mathrm{ad}_{12}\phi = 0 & \forall X \in \mathfrak{m}_1 \\ \mathrm{ad}_{12}(\phi(X))\phi = 0 & \forall X \in \mathfrak{m}_1 \\ \phi \mathrm{ad}_{12}(\phi(X)) = 0 & \forall X \in \mathfrak{m}_1 \\ \phi \mathrm{ad}_{12}\phi = 0 & \forall X \in \mathfrak{m}_2. \end{cases}$$

We begin by computing the first term $\frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D})$. For all $X, Y \in \mathfrak{m}$, we have:

$$\hat{f}^{-1} \circ {}^t\mathrm{ad}(X) \circ \hat{f} \circ \mathrm{ad}(Y) = [(\iota_1 + \iota_2 \circ \phi) \circ \hat{h}^{-1} \circ {}^t(\iota_1 + \iota_2 \circ \phi) + \iota_2 \circ \hat{k}^{-1} \circ {}^t\iota_2] \circ {}^t\mathrm{ad}(X)$$
$$\circ [{}^t\pi_1 \circ \hat{h} \circ \pi_1 + {}^t(\phi \circ \pi_1 - \pi_2) \circ \hat{k} \circ (\phi \circ \pi_1 - \pi_2)] \circ \mathrm{ad}(Y).$$

We can already see from this expression that for each $X, Y \in \mathfrak{m}$, $D(X, Y)$, which is the trace of the above expression, is a polynomial of degree at most 4 in ϕ. The fourth-degree term in $D(X, Y)$ is:

$$\begin{aligned} D^{(4)}(X, Y) &= \operatorname{tr}(\iota_2 \circ \phi \circ \hat{h}^{-1} \circ {}^t\phi \circ {}^t\iota_2 \circ {}^t\mathrm{ad}(X) \circ {}^t\pi_1 \circ {}^t\phi \circ \hat{k} \circ \phi \circ \pi_1 \circ \mathrm{ad}(Y)) \\ &= \operatorname{tr}(\phi \circ \hat{h}^{-1} \circ {}^t\phi \circ {}^t\iota_2 \circ {}^t\mathrm{ad}(X) \circ {}^t\pi_1 \circ {}^t\phi \circ \hat{k} \circ \phi \circ \pi_1 \circ \mathrm{ad}(Y) \circ \iota_2) \\ &= \operatorname{tr}(\phi \circ \hat{h}^{-1} \circ {}^t\phi \circ {}^t\mathrm{ad}_{12}(X) \circ {}^t\phi \circ \hat{k} \circ \phi \circ \mathrm{ad}_{12}(Y)) \\ &= \operatorname{tr}(\hat{h}^{-1} \circ {}^t\phi \circ {}^t\mathrm{ad}_{12}(X) \circ {}^t\phi \circ \hat{k} \circ \phi \circ \mathrm{ad}_{12}(Y) \circ \phi). \end{aligned}$$

On the other hand, $\hat{f}^{-1}$ is a polynomial of degree 2 in ϕ, whose second-degree term is $\iota_2 \circ \phi \circ \hat{h}^{-1} \circ {}^t\phi \circ {}^t\iota_2$.

We deduce therefore that the potential V is a polynomial of degree at most 6 in ϕ, whose sixth-degree term is:

$$\begin{aligned} V^{(6)}(\phi) &= \frac{1}{4}\operatorname{tr}(\iota_2 \circ \phi \circ \hat{h}^{-1} \circ {}^t\phi \circ {}^t\iota_2 \circ \hat{D}^{(4)}) \\ &= \frac{1}{4}\operatorname{tr}(\phi \circ \hat{h}^{-1} \circ {}^t\phi \circ {}^t\iota_2 \circ \hat{D}^{(4)} \circ \iota_2) \\ &= \frac{1}{4}\operatorname{tr}(\hat{h}^{-1} \circ {}^t\phi \circ {}^t\iota_2 \circ \hat{D}^{(4)} \circ \iota_2 \circ \phi) \\ &= \frac{1}{4}\operatorname{tr}(\hat{h}^{-1} \circ \hat{P}^{(6)}(\phi)) \end{aligned}$$

where we have set $\hat{P}^{(6)}(\phi) = {}^t\phi \circ {}^t\iota_2 \circ \hat{D}^{(4)} \circ \iota_2 \circ \phi$.

Thus, we have:

$$V^{(6)}(\phi) = \frac{1}{4}\operatorname{tr}(\hat{h}^{-1} \circ \hat{P}^{(6)}(\phi)).$$

Here also, it is possible to write $V^{(6)}(\phi)$ in a more synthetic way. For this, we make use of the scalar products h and k to define the following scalar products:

$$\begin{aligned} &\text{On } \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2) : \langle u, v\rangle_{hk} = \operatorname{tr}(\hat{h}^{-1} \circ {}^tu \circ \hat{k} \circ v) = \widehat{hk}(u)(v) \\ &\text{On } \mathcal{L}(\mathfrak{m}_1) : \langle u, v\rangle_{hh} = \operatorname{tr}(\hat{h}^{-1} \circ {}^tu \circ \hat{h} \circ v) = \widehat{hh}(u)(v) \\ &\text{On } \mathcal{L}(\mathfrak{m}_2) : \langle u, v\rangle_{kk} = \operatorname{tr}(\hat{k}^{-1} \circ {}^tu \circ \hat{k} \circ v) = \widehat{kk}(u)(v) \end{aligned}$$

which we use to define again the following scalar products:

$$\begin{aligned} &\text{On } \mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2)) : \langle \alpha, \beta\rangle_{hhk} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t\alpha \circ \widehat{hk} \circ \beta) \\ &\text{On } \mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_1)) : \langle \alpha, \beta\rangle_{hhh} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t\alpha \circ \widehat{hh} \circ \beta) \\ &\text{On } \mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_2)) : \langle \alpha, \beta\rangle_{hkk} = \operatorname{tr}(\hat{h}^{-1} \circ {}^t\alpha \circ \widehat{kk} \circ \beta) \\ &\text{On } \mathcal{L}(\mathfrak{m}_2, \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2)) : \langle \alpha, \beta\rangle_{khk} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t\alpha \circ \widehat{hk} \circ \beta) \\ &\text{On } \mathcal{L}(\mathfrak{m}_2, \mathcal{L}(\mathfrak{m}_1)) : \langle \alpha, \beta\rangle_{khh} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t\alpha \circ \widehat{hh} \circ \beta) \\ &\text{On } \mathcal{L}(\mathfrak{m}_2, \mathcal{L}(\mathfrak{m}_2)) : \langle \alpha, \beta\rangle_{kkk} = \operatorname{tr}(\hat{k}^{-1} \circ {}^t\alpha \circ \widehat{kk} \circ \beta). \end{aligned}$$

Using the preceding definitions, we can write:

$$\begin{aligned} P^{(6)}(\phi)(X, Y) &= \operatorname{tr}(\hat{h}^{-1} \circ {}^t\phi \circ {}^t\mathrm{ad}_{12}(\phi(X)) \circ {}^t\phi \circ \hat{k} \circ \phi \circ \mathrm{ad}_{12}(\phi(Y)) \circ \phi) \\ &= \langle \phi \circ \mathrm{ad}_{12}(\phi(X)) \circ \phi, \phi \circ \mathrm{ad}_{12}(\phi(Y)) \circ \phi\rangle_{hk} \\ &= {}^t(\phi \mathrm{ad}_{12}(\phi)\phi)(\langle , \rangle_{hk})(X, Y) \end{aligned}$$

where $\phi\mathrm{ad}_{12}(\phi)\phi$ denotes the element of $\mathcal{L}(\mathfrak{m}_1, \mathcal{L}(\mathfrak{m}_1, \mathfrak{m}_2))$ which associates $\phi \circ \mathrm{ad}_{12}(\phi(X)) \circ \phi$ to each $X \in \mathfrak{m}_1$.

Thus, $\hat{P}^{(6)}(\phi) = {}^t(\phi\mathrm{ad}_{12}(\phi)\phi) \circ \widehat{hk} \circ (\phi\mathrm{ad}_{12}(\phi)\phi)$, therefore

$$\begin{aligned} \operatorname{tr}(\hat{h}^{-1} \circ \hat{P}^{(6)}(\phi)) &= \operatorname{tr}(\hat{h}^{-1} \circ {}^t(\phi\mathrm{ad}_{12}(\phi)\phi) \circ \widehat{hk} \circ (\phi\mathrm{ad}_{12}(\phi)\phi)) \\ &= \langle \phi\mathrm{ad}_{12}(\phi)\phi, \phi\mathrm{ad}_{12}(\phi)\phi\rangle_{hhk}. \end{aligned}$$

Thus, we have

$$V^{(6)}(\phi) = \frac{1}{4}\|\phi \mathrm{ad}_{12}(\phi)\phi\|^2_{hhk}$$

which proves the positivity of this term.

Moreover, if we denote by $\Gamma^{(6)}$ the set of zeroes of $V^{(6)}$, we see immediately that:

$$\Gamma^{(6)} = \{\phi \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)/\phi \circ \mathrm{ad}_{12}(\phi(X)) \circ \phi = 0\ \forall X \in \mathfrak{m}_1\}.$$

Thus, we have a condition of the third-degree in ϕ, and if this condition is verified for every ϕ that is, if $\Gamma^{(6)} = \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$, then $V^{(6)} = 0$.

Because of the positivity of $\frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D})$, V cannot be of degree 5, therefore it becomes a polynomial of degree 4 at most, whose quartic terms can be computed in a similar manner to give:

$$V^{(4)}(\phi) = \frac{1}{4}\|\phi \mathrm{ad}_{11}(\phi) - \mathrm{ad}_{22}(\phi)\phi + \phi \mathrm{ad}_{12}\phi\|^2_{hhk} + \frac{1}{4}\|\mathrm{ad}_{12}(\phi)\phi\|^2_{hhh} + \frac{1}{4}\|\phi \mathrm{ad}_{12}(\phi)\|^2_{hkk} + \frac{1}{4}\|\phi \mathrm{ad}_{12}\phi\|^2_{khk}$$

which proves the positivity of $V^{(4)}$.

Moreover, if we denote by $\Gamma^{(4)}$ the set of zeroes of $V^{(4)}$, we see immediately that $\Gamma^{(4)}$ is the set of the elements ϕ in $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ that satisfy the quadratic conditions (C).

If these conditions are verified for every ϕ that is, if $\Gamma^{(4)} = \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$, then $V^{(4)} = 0$.

Because of the positivity of $\frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D})$, V cannot be of degree 3, therefore it becomes a polynomial of degree 2 at most. It is also possible to compute the quadratic terms that appear in $\frac{1}{4}\mathrm{tr}(\hat{f}^{-1} \circ \hat{D})$, which are necessarily positive, but to get all the quadratic contributions in V, one has to include also the quadratic terms coming from $\frac{1}{2}\mathrm{tr}(\hat{f}^{-1} \circ \hat{B})$ and $\mathrm{tr}(\hat{f}^{-1} \circ \hat{D}^{\mathfrak{h}})$, and these have no reason to be of fixed sign.

Proposition 3.4. *If the elements h and k, fixed respectively in $S_2^{H++}(\mathfrak{m}_1)$ and $S_2^{H++}(\mathfrak{m}_2)$, are chosen* Ad$(N(H))$-*invariant, then the potential $V(\phi)$ is invariant under the action of the group $N(H)|H$.*

Proof. The proof is similar to that of Proposition 2.2. □

Let us take the special case in which $\mathfrak{m}_2$ is a Lie algebra. This case covers the known situations in which we have Higgs fields as equivariant maps between two representation spaces. If $\mathfrak{m}_2$ is a Lie algebra, then ad_{12} vanishes identically, and therefore $V^{(6)}(\phi)$ is zero. But in this case, we can verify that $V^{(5)}(\phi)$ will also be zero. Thus, we are left with a quartic potential which is exactly what we have in the known situations.

3.4. The Yang–Mills term and the kinetic term

As outlined in the first paragraph of Section 2.2, the Lagrangian of the multidimensional G-invariant gravity theory reduces, after integration on the internal variables, to the Lagrangian of an Einstein–Yang–Mills theory coupled to a scalar field f. We will begin by writing an intrinsic expression of this reduced Lagrangian, and then explain the notations used while investigating each term

$$\mathcal{L}(\gamma, \alpha, f)(x) = -V(\gamma)(x) - V(\Omega(q)) + K(f(q)) - V(f(q)) \quad \text{where } q \in Q_x$$

where V stands for "potential" and K for "kinetic". Explicitly,

$$\mathcal{L}(\gamma, \alpha, f)(x) = \rho_M(\gamma)(x) - \frac{1}{2}\|\bar{\Omega}(q)\|^2_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)} + \frac{1}{2}\|D\hat{f}(q)\|^2_{\tilde{\gamma}(q)f(q)f^*(q)} + \rho_{G/H}(f(q))$$

where ρ means "scalar curvature functional". The term $\rho_{G/H}(f(q))$ is the opposite of the potential for the Thiry scalar field f, and has already been discussed in Sections 2.2 and 3.3. $\rho_M(\gamma)(x)$ is the Lagrangian of the gravity sector in M, and we'll have nothing more to say about it. The remainder of this section will be devoted to the two middle terms.

- The Yang–Mills term $V(\Omega(q)) = \frac{1}{2}\|\bar{\Omega}(q)\|^2_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)}$.

  Ω denotes the curvature two-form $D\alpha$ of the connection α. Let $\mathfrak{k}$ be the Lie algebra of the gauge group $N(H)|H$. Then $\Omega(q)$ is a $\mathfrak{k}$-valued antisymmetric bilinear form on T_qQ, and by horizontality of Ω, we can consider only its restriction to the horizontal subspace $Z_q = \operatorname{Ker}\alpha(q)$. We denote by $\bar{\Omega}(q) : Z_q \longrightarrow \mathcal{L}(Z_q, \mathfrak{k})$ the linear map canonically associated to $\Omega(q)_{|Z_q \times Z_q}$. Let $\tilde{\gamma}(q)$ be the scalar product on Z_q obtained by horizontal lift of $\gamma(x)$, and $\bar{\gamma}(q) : Z_q \longrightarrow Z_q^*$ the linear map canonically associated to $\tilde{\gamma}(q)$. Finally, let $\tilde{f}(q)$ be the restriction to $\mathfrak{k}$ of $f(q)$, and $\bar{f}(q) : \mathfrak{k} \longrightarrow \mathfrak{k}^*$ the linear map canonically associated to $\tilde{f}(q)$.

Similarly to what we have done when expressing the potential of the scalar fields, we introduce the scalar product:

$$\text{On } \mathcal{L}(Z_q, \mathfrak{k}) : \langle u, v\rangle_{\tilde{\gamma}(q)\tilde{f}(q)} = \operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t u \circ \bar{f}(q) \circ v) = \overline{\gamma(q) f(q)}(u)(v)$$

which we use to define again the following scalar product:

$$\text{On } \mathcal{L}(Z_q, \mathcal{L}(Z_q, \mathfrak{k})) : \langle \alpha, \beta\rangle_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)} = \operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\alpha \circ \overline{\gamma(q) f(q)} \circ \beta).$$

Now we have introduced all the notations that give sense to the expression:

$$V(\Omega(q)) = \frac{1}{2}\|\bar{\Omega}(q)\|^2_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)}$$

but let us write it also in the expanded way:

$$\begin{aligned}
V(\Omega(q)) &= \frac{1}{2}\langle \bar{\Omega}(q), \bar{\Omega}(q)\rangle_{\tilde{\gamma}(q)\tilde{\gamma}(q)\tilde{f}(q)}\\
&= \frac{1}{2}\operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}(q) \circ \overline{\gamma(q) f(q)} \circ \bar{\Omega}(q))\\
&= \frac{1}{2}\operatorname{tr}(\bar{\gamma}(q)^{-1} \circ \bar{D})
\end{aligned}$$

where $\bar{D} = {}^t\bar{\Omega}(q) \circ \overline{\gamma(q) f(q)} \circ \bar{\Omega}(q)$ is a linear map from Z_q to Z_q^*, to which is associated the following bilinear form on Z_q:

$$\begin{aligned}
D(X, Y) &= {}^t\bar{\Omega}(q)(\langle,\rangle_{\tilde{\gamma}(q)\tilde{f}(q)})(X, Y)\\
&= \langle {}^t\bar{\Omega}(q)(X), {}^t\bar{\Omega}(q)(Y)\rangle_{\tilde{\gamma}(q)\tilde{f}(q)}\\
&= \operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}(q)(X) \circ \bar{f}(q) \circ \bar{\Omega}(q)(Y))
\end{aligned}$$

for every $X, Y \in Z_q$.

Our next purpose is to expand this Yang–Mills term with respect to our decomposition of f in terms of h, k, ϕ. For this, we set $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{m}_1$ and $\mathfrak{k}_2 = \mathfrak{k} \cap \mathfrak{m}_2$, so that $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$. Let $\bar{\pi}_1 : \mathfrak{k} \longrightarrow \mathfrak{k}_1$ and $\bar{\pi}_2 : \mathfrak{k} \longrightarrow \mathfrak{k}_2$ be the corresponding projectors. We define:

. $\bar{\phi}(q) = -\mathrm{pr}^{\mathfrak{k}_2}_{|\mathfrak{k}_1}$, where $\mathrm{pr}^{\mathfrak{k}_2}$ is the orthogonal projector on $\mathfrak{k}_2$ for the Euclidean structure defined by $\tilde{f}(q)$.
. $\forall X_1, Y_1 \in \mathfrak{k}_1, \tilde{h}(q)(X_1, Y_1) = \tilde{f}(q)(X_1 + \bar{\phi}(q)(X_1), Y_1 + \bar{\phi}(q)(Y_1))$
. $\tilde{k} = \tilde{f}_{|\mathfrak{k}_2 \times \mathfrak{k}_2}$
so that $\tilde{h}(q) \in S_2(\mathfrak{k}_1)$, $\tilde{k}(q) \in S_2(\mathfrak{k}_2)$, $\bar{\phi}(q) \in \mathcal{L}(\mathfrak{k}_1, \mathfrak{k}_2)$, and:

$$\bar{f}(q) = {}^t\bar{\pi}_1 \circ \bar{h}(q) \circ \bar{\pi}_1 + {}^t(\bar{\phi}(q) \circ \bar{\pi}_1 - \bar{\pi}_2) \circ \bar{k}(q) \circ (\bar{\phi}(q) \circ \bar{\pi}_1 - \bar{\pi}_2).$$

Finally, we set:

$$\begin{aligned}
\bar{\Omega}_1(q)(X) &= \bar{\pi}_1 \circ \bar{\Omega}(q)(X) \in \mathcal{L}(Z_q, \mathfrak{k}_1)\\
\bar{\Omega}_2(q)(X) &= \bar{\pi}_2 \circ \bar{\Omega}(q)(X) \in \mathcal{L}(Z_q, \mathfrak{k}_2).
\end{aligned}$$

With all these definitions, we have:

$$\begin{aligned}
D(X, Y) &= \operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_1(q)(X) \circ \bar{h}(q) \circ \bar{\Omega}_1(q)(Y))\\
&= +\operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_2(q)(X) \circ \bar{k}(q) \circ \bar{\Omega}_2(q)(Y))\\
&= +\operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_1(q)(X) \circ {}^t\bar{\phi}(q) \circ \bar{k}(q) \circ \bar{\phi}(q) \circ \bar{\Omega}_1(q)(Y))\\
&= -\operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_1(q)(X) \circ {}^t\bar{\phi}(q) \circ \bar{k}(q) \circ \bar{\Omega}_2(q)(Y))\\
&= -\operatorname{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\bar{\Omega}_2(q)(X) \circ \bar{k}(q) \circ \bar{\phi}(q) \circ \bar{\Omega}_1(q)(Y)).
\end{aligned}$$

We can state now the following theorem:

Theorem 3.5. *If the trivial representation of H lies completely either in $\mathfrak{m}_1$ or in $\mathfrak{m}_2$, then there is no direct coupling between the scalar fields ϕ and the Yang–Mills strength Ω, and we have:*

$$V(\Omega(q)) = \frac{1}{2}\|\bar{\Omega}_1(q)\|^2_{\bar{\gamma}(q)\bar{\gamma}(q)\tilde{h}(q)} + \frac{1}{2}\|\bar{\Omega}_2(q)\|^2_{\bar{\gamma}(q)\bar{\gamma}(q)\tilde{k}(q)}.$$

Proof. It is not difficult to see that $\mathfrak{k}$ carries the trivial representation of H (cf. [9] for example). The assumption of the theorem implies then that at least one of the two subspaces $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is zero. Therefore, we have $\mathcal{L}(\mathfrak{k}_1, \mathfrak{k}_2) = 0$. We deduce then that in the last expression of $D(X, Y)$, we have $\bar{\phi} = 0$, and therefore we are left only with the first two terms. □

- The kinetic term $K(f(q)) = \frac{1}{2}\|\mathcal{D}\hat{f}(q)\|^2_{\bar{\gamma}(q)f(q)f^*(q)}$.

We shall follow the same steps as for the Yang–Mills term. $\mathcal{D}f$ denotes the covariant derivative of f, therefore $\mathcal{D}f(q)$ is a linear map from T_qQ to $S^2_H(\mathfrak{m})$. By horizontality of $\mathcal{D}f$, we can consider only the restriction of $\mathcal{D}f(q)$ to the horizontal subspace Z_q, and in fact we shall work with $\mathcal{D}\hat{f}(q) : Z_q \longrightarrow \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*)$. For every $X \in Z_q$, we have:

$$\begin{aligned}\mathcal{D}\hat{f}(q)(X) &= \mathrm{d}\hat{f}(q)(X) + \rho'_{\alpha(q)(X)}(\hat{f}(q)) \\ &= \mathrm{d}\hat{f}(q)(X) - ({}^t\mathrm{ad}_{\alpha(q)(X)} \circ \hat{f}(q)) - (\hat{f}(q) \circ \mathrm{ad}_{\alpha(q)(X)})\end{aligned}$$

(see the remark before Proposition 2.2). $\bar{\gamma}(q)$ has been defined in the previous paragraph.

In the same spirit, we introduce the following scalar product:

$$\text{On } \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*) : \langle u, v\rangle_{f(q)f^*(q)} = \mathrm{tr}(\hat{f}(q)^{-1} \circ {}^tu \circ {}^t\hat{f}(q)^{-1} \circ v) = f\widehat{(q)f^*}(q)(u)(v)$$

which we use to define again the following scalar product:

$$\text{On } \mathcal{L}(Z_q, \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*)) : \langle \alpha, \beta\rangle_{\bar{\gamma}(q)f(q)f^*(q)} = \mathrm{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\alpha \circ f\widehat{(q)f^*}(q) \circ \beta)$$

($f\widehat{(q)f^*}(q) : \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*) \longrightarrow \mathcal{L}^H(\mathfrak{m}, \mathfrak{m}^*)^*$ is the isomorphism canonically associated to $\langle , \rangle_{f(q)f^*(q)}$).

Now we have introduced all the notations that give sense to the expression:

$$K(f(q)) = \frac{1}{2}\|\mathcal{D}\hat{f}(q)\|^2_{\bar{\gamma}(q)f(q)f^*(q)}$$

but let us write it also in the expanded way:

$$\begin{aligned}K(f(q)) &= \frac{1}{2}\langle \mathcal{D}\hat{f}(q), \mathcal{D}\hat{f}(q)\rangle_{\bar{\gamma}(q)f(q)f^*(q)} \\ &= \frac{1}{2}\mathrm{tr}(\bar{\gamma}(q)^{-1} \circ {}^t\mathcal{D}\,\hat{f}(q) \circ f\widehat{(q)f^*}(q) \circ \mathcal{D}\hat{f}(q)) \\ &= \frac{1}{2}\mathrm{tr}(\bar{\gamma}(q)^{-1} \circ \bar{E})\end{aligned}$$

where $\bar{E} = {}^t\mathcal{D}\,\hat{f}(q) \circ f\widehat{(q)f^*}(q) \circ \mathcal{D}\hat{f}(q)$ is a linear map from Z_q to Z_q^*, to which is associated the following bilinear form on Z_q:

$$\begin{aligned}E(X, Y) &= {}^t\mathcal{D}\,\hat{f}(q)(\langle , \rangle_{f(q)f^*(q)})(X, Y) \\ &= \langle {}^t\mathcal{D}\,\hat{f}(q)(X), {}^t\mathcal{D}\,\hat{f}(q)(Y)\rangle_{f(q)f^*(q)} \\ &= \mathrm{tr}(\hat{f}(q)^{-1} \circ {}^t\mathcal{D}\,\hat{f}(q)(X) \circ {}^t\hat{f}(q)^{-1} \circ \mathcal{D}\hat{f}(q)(Y))\end{aligned}$$

for every $X, Y \in Z_q$.

Again, our next purpose is to expand this kinetic term with respect to our decomposition of f in terms of h, k, ϕ. For this, we recall that:

$$\hat{f}(q) = {}^t\pi_1 \circ \hat{h}(q) \circ \pi_1 + {}^t(\phi(q) \circ \pi_1 - \pi_2) \circ \hat{k}(q) \circ (\phi(q) \circ \pi_1 - \pi_2)$$

and

$$\hat{f}(q)^{-1} = (\iota_1 + \iota_2 \circ \phi(q)) \circ \hat{h}(q)^{-1} \circ {}^t(\iota_1 + \iota_2 \circ \phi(q)) + \iota_2 \circ \hat{k}(q)^{-1} \circ {}^t\iota_2$$

and we write ${}^t\hat{f}(q)^{-1}$. Then, we compute the covariant derivatives $\mathcal{D}\hat{f}(q)(Y)$ and ${}^t\mathcal{D}\hat{f}(q)(X)$ with the usual Leibnitz rule for products, getting the sum of nine terms. Notice for example that:

$$\begin{aligned}
&\mathcal{D}({}^t\pi_1 \circ \hat{h}(q) \circ \pi_1)(X) \\
&\quad = {}^t\pi_1 \circ \mathrm{d}\hat{h}(q)(X) \circ \pi_1 - ({}^t\mathrm{ad}_{\alpha(q)(X)} \circ {}^t\pi_1 \circ \hat{h}(q) \circ \pi_1) - ({}^t\pi_1 \circ \hat{h}(q) \circ \pi_1 \circ \mathrm{ad}_{\alpha(q)(X)}) \\
&\quad = {}^t\pi_1 \circ \mathrm{d}\hat{h}(q)(X) \circ \pi_1 - ({}^t\pi_1 \circ {}^t\mathrm{ad}^{(1)}_{\alpha(q)(X)} \circ \hat{h}(q) \circ \pi_1) - ({}^t\pi_1 \circ \hat{h}(q) \circ \mathrm{ad}^{(1)}_{\alpha(q)(X)} \circ \pi_1) \\
&\quad = {}^t\pi_1 \circ [\mathrm{d}\hat{h}(q)(X) - ({}^t\mathrm{ad}^{(1)}_{\alpha(q)(X)} \circ \hat{h}(q)) - (\hat{h}(q) \circ \mathrm{ad}^{(1)}_{\alpha(q)(X)})] \circ \pi_1 \\
&\quad = {}^t\pi_1 \circ \mathcal{D}\hat{h}(q)(X) \circ \pi_1
\end{aligned}$$

where $\mathcal{D}\hat{h}(q)$ is naturally defined by:

$$\begin{aligned}
\mathcal{D}\hat{h}(q)(X) &= \mathrm{d}\hat{h}(q)(X) - ({}^t\mathrm{ad}^{(1)}_{\alpha(q)(X)} \circ \hat{h}(q)) - (\hat{h}(q) \circ \mathrm{ad}^{(1)}_{\alpha(q)(X)}) \\
&= \mathrm{d}\hat{h}(q)(X) + \eta^1_{\alpha(q)(X)}(\hat{h}(q)).
\end{aligned}$$

Similarly,

$$\mathcal{D}({}^t\pi_2 \circ \hat{k}(q) \circ \pi_2)(X) = {}^t\pi_2 \circ \mathcal{D}\hat{k}(q)(X) \circ \pi_2$$

where $\mathcal{D}\hat{k}(q)$ is naturally defined by:

$$\begin{aligned}
\mathcal{D}\hat{k}(q)(X) &= \mathrm{d}\hat{k}(q)(X) - ({}^t\mathrm{ad}^{(2)}_{\alpha(q)(X)} \circ \hat{k}(q)) - (\hat{k}(q) \circ \mathrm{ad}^{(2)}_{\alpha(q)(X)}) \\
&= \mathrm{d}\hat{k}(q)(X) + \eta^2_{\alpha(q)(X)}(\hat{k}(q)).
\end{aligned}$$

As for $\mathcal{D}\phi(q)$, it is naturally defined by:

$$\begin{aligned}
\mathcal{D}\phi(q)(X) &= \mathrm{d}\phi(q)(X) + \eta^0_{\alpha(q)(X)}(\phi(q)) \\
&= \mathrm{d}\phi(q)(X) + (\mathrm{ad}^{(2)}_{\alpha(q)(X)} \circ \phi(q)) - (\phi(q) \circ \mathrm{ad}^{(1)}_{\alpha(q)(X)}).
\end{aligned}$$

Finally, we replace in the expression of $E(X, Y)$ and expand! The long calculation gives a large number of terms, between which we discover happily an equally large number of cancellations, and we are left finally with the terms:

$$\|\mathcal{D}\phi(q)\|^2_{h(q)h(q)k(q)} + \text{four terms depending on } \mathcal{D}k$$

Thus, we can state the following result:

Theorem 3.6. *If, in the decomposition of f in terms of h, k, ϕ, the matter field k is taken constant, then the kinetic term of the scalar field ϕ is proportional to the natural term: $\frac{1}{2}\|\mathcal{D}\phi(q)\|^2_{h(q)h(q)k(q)}$.*

This result agrees with that of [5], where the field k was taken constant: the same kinetic term is obtained for ϕ.

We conclude this section by checking the gauge invariance of the theory:

Proposition 3.7. *If we freeze the scalar fields $h : Q \longrightarrow S_2^H(\mathfrak{m}_1)$ and $k : Q \longrightarrow S_2^H(\mathfrak{m}_2)$ by taking them constant, the value of each being $\mathrm{Ad}(N(H))$-invariant and positive definite, and if the trivial representation of H lies completely either in $\mathfrak{m}_1$ or in $\mathfrak{m}_2$, then the Lagrangian $\mathcal{L}(\gamma, \alpha, \phi)$ is invariant under the action of the group $\mathcal{G} = C^\infty_{N(H)|H}(Q, N(H)|H)$ of gauge transformations.*

Proof. We already mentioned the invariance of the potential $V(\phi)$ in Proposition 3.4. The gravity sector in M brings no problem. We are left with the Yang–Mills term of Theorem 3.5 and the natural kinetic term of Theorem 3.6. And these are easily checked to be invariant when a gauge transformation $n : Q \longrightarrow N(H)|H$ acts on all the fields. □

4. Example with $G = SU(5)$, $H \simeq U(1)$ and $N(H)|H \simeq \mathbb{T}^3$

We begin by setting some notations that we are going to use in the following.

- For each $n \in \mathbb{N}^*$, $\mathbb{T}^n$ denotes the n-dimensional torus $U(1) \times \cdots \times U(1)$.
- For each $p \in \mathbb{Z}$, $\mathbb{C}_p$ denotes the vector space $\mathbb{C}$ endowed with the irreducible representation of $U(1)$ labeled by the integer p.

$U(1)$ is embedded in $SU(5)$ by the following injective homomorphism of Lie groups:
$\lambda : U(1) \longrightarrow SU(5)$ defined by:

$$\lambda(e^{i\theta}) = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & e^{i2\theta} & 0 & 0 & 0 \\ 0 & 0 & e^{i3\theta} & 0 & 0 \\ 0 & 0 & 0 & e^{i4\theta} & 0 \\ 0 & 0 & 0 & 0 & e^{-i10\theta} \end{pmatrix}.$$

We set $H = \lambda(U(1))$. The normalizer of H in G is then:

$$N(H) = \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & e^{i\theta_4} & 0 \\ 0 & 0 & 0 & 0 & e^{-i(\theta_1+\theta_2+\theta_3+\theta_4)} \end{pmatrix} ; \theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R} \right\} \simeq \mathbb{T}^4.$$

First, we study the adjoint action of H on $\mathfrak{g} = \mathfrak{su}(5)$. Let $h = \lambda(e^{i\theta}) \in H$, and

$$A = \begin{pmatrix} iy_{11} & -\bar{z}_{21} & z_{13} & z_{14} & z_{15} \\ z_{21} & iy_{22} & -\bar{z}_{32} & z_{24} & z_{25} \\ -\bar{z}_{13} & z_{32} & iy_{33} & -\bar{z}_{43} & z_{35} \\ -\bar{z}_{14} & -\bar{z}_{24} & z_{43} & iy_{44} & z_{45} \\ -\bar{z}_{15} & -\bar{z}_{25} & -\bar{z}_{35} & -\bar{z}_{45} & -i(y_{11}+y_{22}+y_{33}+y_{44}) \end{pmatrix} \in \mathfrak{su}(5).$$

Then:

$$hAh^{-1} = \begin{pmatrix} iy_{11} & -e^{-i\theta}\bar{z}_{21} & e^{-i2\theta}z_{13} & e^{-i3\theta}z_{14} & e^{i11\theta}z_{15} \\ e^{i\theta}z_{21} & iy_{22} & -e^{-i\theta}\bar{z}_{32} & e^{-i2\theta}z_{24} & e^{i12\theta}z_{25} \\ -e^{i2\theta}\bar{z}_{13} & e^{i\theta}z_{32} & iy_{33} & -e^{-i\theta}\bar{z}_{43} & e^{i13\theta}z_{35} \\ -e^{i3\theta}\bar{z}_{14} & -e^{i2\theta}\bar{z}_{24} & e^{i\theta}z_{43} & iy_{44} & e^{i14\theta}z_{45} \\ -e^{-i11\theta}\bar{z}_{15} & -e^{-i12\theta}\bar{z}_{25} & -e^{-i13\theta}\bar{z}_{35} & -e^{-i14\theta}\bar{z}_{45} & -i(y_{11}+y_{22}+y_{33}+y_{44}) \end{pmatrix}.$$

We deduce the decomposition of $\mathfrak{g} = \mathfrak{su}(5)$ under the adjoint action of H:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}} \oplus \overset{1}{\mathfrak{l}_{43}} \oplus \overset{-2}{\mathfrak{l}_{13}} \oplus \overset{-2}{\mathfrak{l}_{24}} \oplus \overset{-3}{\mathfrak{l}_{14}} \oplus \overset{11}{\mathfrak{l}_{15}} \oplus \overset{12}{\mathfrak{l}_{25}} \oplus \overset{13}{\mathfrak{l}_{35}} \oplus \overset{14}{\mathfrak{l}_{45}}$$

where:

- $$\mathfrak{h} = \left\{ \begin{pmatrix} i\theta & 0 & 0 & 0 & 0 \\ 0 & i2\theta & 0 & 0 & 0 \\ 0 & 0 & i3\theta & 0 & 0 \\ 0 & 0 & 0 & i4\theta & 0 \\ 0 & 0 & 0 & 0 & -i10\theta \end{pmatrix} ; \theta \in \mathbb{R} \right\} \simeq \mathrm{Lie}(H) \simeq \mathfrak{u}(1) \simeq i\mathbb{R}$$

- $$\mathfrak{k} = \left\{ \begin{pmatrix} ix_1 & 0 & 0 & 0 & 0 \\ 0 & ix_2 & 0 & 0 & 0 \\ 0 & 0 & ix_3 & 0 & 0 \\ 0 & 0 & 0 & -i(x_1+x_2+x_3) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; x_1, x_2, x_3 \in \mathbb{R} \right\} \simeq \mathrm{Lie}(N(H)|H) \simeq \mathrm{Lie}(\mathbb{T}^3) \simeq \mathbb{R}^3.$$

- $$\text{For } 1 \le j < k \le 5, \quad \mathfrak{l}_{jk} = \left\{ \begin{pmatrix} & z \\ -\bar{z} & \end{pmatrix}; z \in \mathbb{C} \right\} \simeq \mathbb{C}.$$
- $$\text{For } 1 \le k < j \le 5, \quad \mathfrak{l}_{jk} = \left\{ \begin{pmatrix} & -\bar{z} \\ z & \end{pmatrix}; z \in \mathbb{C} \right\} \simeq \mathbb{C}$$

(in the last two matrices, z is the (j,k)-entry and $-\bar{z}$ is the (k,j)-entry).

$\mathfrak{h}$ and $\mathfrak{k}$ carry the trivial representation of H, and for $1 \le j < k \le 5$, $\overset{p}{\mathfrak{l}_{jk}} \simeq \mathbb{C}_p$.

Second, we study the adjoint action of $N(H)$ on $\mathfrak{g} = \mathfrak{su}(5)$. Noticing that $N(H) \simeq \mathbb{T}^4$ is nothing but the natural maximal torus of $\mathfrak{su}(5)$, we see immediately that the decomposition of $\mathfrak{g}$ under $N(H)$ reads:

$$\begin{aligned} \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \overset{(-1,1,0,0)}{\mathfrak{l}_{21}} \oplus \overset{(0,-1,1,0)}{\mathfrak{l}_{32}} \oplus \overset{(0,0,-1,1)}{\mathfrak{l}_{43}} \oplus \overset{(1,0,-1,0)}{\mathfrak{l}_{13}} \oplus \overset{(0,1,0,-1)}{\mathfrak{l}_{24}} \oplus \overset{(1,0,0,-1)}{\mathfrak{l}_{14}} \\ \oplus \overset{(2,1,1,1)}{\mathfrak{l}_{15}} \oplus \overset{(1,2,1,1)}{\mathfrak{l}_{25}} \oplus \overset{(1,1,2,1)}{\mathfrak{l}_{35}} \oplus \overset{(1,1,1,2)}{\mathfrak{l}_{45}} \end{aligned}$$

where $\overset{(p_1,p_2,p_3,p_4)}{\mathfrak{l}_{jk}} \simeq \mathbb{C}$ carrying the irreducible representation of $\mathbb{T}^4$ labeled by the element $(p_1, p_2, p_3, p_4) \in \mathbb{Z}^4$.

Let us define the two following subspaces of $\mathfrak{g}$:

$$\begin{aligned} \mathfrak{m}_1 &= \mathfrak{k} \oplus \overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}} \oplus \overset{-2}{\mathfrak{l}_{13}} \\ \mathfrak{m}_2 &= \overset{-3}{\mathfrak{l}_{14}} \oplus \overset{1}{\mathfrak{l}_{43}} \oplus \overset{-2}{\mathfrak{l}_{24}} \oplus \overset{11}{\mathfrak{l}_{15}} \oplus \overset{12}{\mathfrak{l}_{25}} \oplus \overset{13}{\mathfrak{l}_{35}} \oplus \overset{14}{\mathfrak{l}_{45}}. \end{aligned}$$

It is clear that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $\mathrm{Ad}(N(H))$-invariant complementary subspaces of $\mathfrak{m}$, and thus we have an $\mathrm{Ad}(N(H))$-invariant splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We deduce a space of Higgs fields:

$$\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = \mathcal{L}(\overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}}, \overset{1}{\mathfrak{l}_{43}}) \oplus \mathcal{L}(\overset{-2}{\mathfrak{l}_{13}}, \overset{-2}{\mathfrak{l}_{24}})$$

It is not difficult to see that $\dim \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = 6$ (on $\mathbb{R}$).

A natural basis for the vector space $\mathfrak{m}$ is given by the following matrices:

$$\begin{pmatrix} \mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(basis for } \mathfrak{k}\text{);}$$

$$\text{for } 1 \le j < k \le 5, \quad a_{jk} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \qquad b_{jk} = \begin{pmatrix} & \mathrm{i} \\ \mathrm{i} & \end{pmatrix} \quad \text{(basis for } \mathfrak{l}_{jk}\text{).}$$

$$\text{for } 1 \le k < j \le 5, \quad a_{jk} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad b_{jk} = \begin{pmatrix} & \mathrm{i} \\ \mathrm{i} & \end{pmatrix} \quad \text{(basis for } \mathfrak{l}_{jk}\text{).}$$

(In a_{jk}, 1 is the (j,k)-entry and -1 is the (k,j)-entry, etc.)

Let $\phi \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. In fact, ϕ is completely determined by its restriction (also denoted by ϕ):

$$\phi : \overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}} \oplus \overset{-2}{\mathfrak{l}_{13}} \longrightarrow \overset{1}{\mathfrak{l}_{43}} \oplus \overset{-2}{\mathfrak{l}_{24}}.$$

We set $\mathfrak{l}_1 = \overset{1}{\mathfrak{l}_{21}} \oplus \overset{1}{\mathfrak{l}_{32}} \oplus \overset{-2}{\mathfrak{l}_{13}}$, $\mathfrak{l}_2 = \overset{1}{\mathfrak{l}_{43}} \oplus \overset{-2}{\mathfrak{l}_{24}}$, and define the six real numbers $\varphi_{21}, \psi_{21}, \varphi_{32}, \psi_{32}, \varphi_{13}, \psi_{13}$ by:

$$\begin{aligned} \phi(a_{21}) &= \varphi_{21} a_{43} + \psi_{21} b_{43} \qquad && (\text{and } \phi(b_{21}) = -\psi_{21} a_{43} + \varphi_{21} b_{43}) \\ \phi(a_{32}) &= \varphi_{32} a_{43} + \psi_{32} b_{43} \qquad && (\text{and } \phi(b_{32}) = -\psi_{32} a_{43} + \varphi_{32} b_{43}) \\ \phi(a_{13}) &= \varphi_{13} a_{24} + \psi_{13} b_{24} \qquad && (\text{and } \phi(b_{13}) = -\psi_{13} a_{24} + \varphi_{13} b_{24}). \end{aligned}$$

The real numbers φ_{21}, ψ_{21}, φ_{32}, ψ_{32}, φ_{13}, ψ_{13} constitute the six components of the Higgs field ϕ in the bases of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ defined above.

We shall need the three complex components of ϕ, defined by:

$$\phi_{21} = \varphi_{21} + \mathrm{i}\psi_{21}, \qquad \phi_{32} = \varphi_{32} + \mathrm{i}\psi_{32}, \qquad \phi_{13} = \varphi_{21} + \mathrm{i}\psi_{13}.$$

$$\text{If } A = \begin{pmatrix} 0 & -\bar{z}_{21} & z_{13} & 0 & 0 \\ z_{21} & 0 & -\bar{z}_{32} & 0 & 0 \\ -\bar{z}_{13} & z_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{l}_1,$$

$$\text{then } \phi(A) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{13}z_{13} & 0 \\ 0 & 0 & 0 & -\bar{\phi}_{21}\bar{z}_{21} - \bar{\phi}_{32}\bar{z}_{32} & 0 \\ 0 & -\bar{\phi}_{13}\bar{z}_{13} & \phi_{21}z_{21} + \phi_{32}z_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{l}_2.$$

Let us study the representation of $N(H)|H$ on $\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2)$. For this, we first look at the representation of $N(H)$ on $\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2)$:

Given

$$P = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta_2} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\theta_3} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\mathrm{i}\theta_4} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{-\mathrm{i}(\theta_1+\theta_2+\theta_3+\theta_4)} \end{pmatrix} \in N(H),$$

we have

$$\mathrm{Ad}_{P^{-1}}(A) = P^{-1}AP = \begin{pmatrix} 0 & -\mathrm{e}^{\mathrm{i}(-\theta_1+\theta_2)}\bar{z}_{21} & \mathrm{e}^{\mathrm{i}(-\theta_1+\theta_3)}z_{13} & 0 & 0 \\ \mathrm{e}^{-\mathrm{i}(-\theta_1+\theta_2)}z_{21} & 0 & -\mathrm{e}^{\mathrm{i}(-\theta_2+\theta_3)}\bar{z}_{32} & 0 & 0 \\ -\mathrm{e}^{-\mathrm{i}(-\theta_1+\theta_3)}\bar{z}_{13} & \mathrm{e}^{-\mathrm{i}(-\theta_2+\theta_3)}z_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad \mathrm{Ad}_P \circ \phi \circ \mathrm{Ad}_{P^{-1}}(A) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\bar{\beta} & 0 \\ 0 & -\bar{\alpha} & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\alpha = \mathrm{e}^{\mathrm{i}(\theta_2-\theta_4-\theta_1+\theta_3)}\phi_{13}z_{13}$ and $\beta = \mathrm{e}^{-\mathrm{i}(\theta_3-\theta_4-\theta_1+\theta_2)}\phi_{21}z_{21} + \mathrm{e}^{-\mathrm{i}(\theta_3-\theta_4-\theta_2+\theta_3)}\phi_{32}z_{32}$.

This shows that ϕ_{13}, ϕ_{21} and ϕ_{32} transform respectively according to the representations $(-1, 1, 1, -1)$, $(1, -1, -1, 1)$ and $(0, 1, -2, 1)$ of $\mathbb{T}^4 \simeq N(H)$.

Now we turn to the action of $N(H)|H$. First, we define an homomorphism of Lie groups $f : N(H) \longrightarrow G$ by setting:

$$f(P) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(4\theta_1-\theta_4)} & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(3\theta_1-\theta_3)} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}(2\theta_1-\theta_2)} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\mathrm{i}(9\theta_1-\theta_2-\theta_3-\theta_4)} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next, we set:

$$K = \left\{ \begin{pmatrix} e^{ix_1} & 0 & 0 & 0 & 0 \\ 0 & e^{ix_2} & 0 & 0 & 0 \\ 0 & 0 & e^{ix_3} & 0 & 0 \\ 0 & 0 & 0 & e^{-i(x_1+x_2+x_3)} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

It is not difficult to seethat $\mathrm{Ker} f = H$ and $\mathrm{Im} f = K$. Therefore, we have a surjective homomorphism that we also denote by f, and this $f : N(H) \longrightarrow K$ induces an isomorphism $\bar{f} : N(H)|H \longrightarrow K$. Thus, $N(H)|H \simeq K \simeq \mathbb{T}^3$.

Now if $\rho : N(H) \longrightarrow GL(\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2))$ is the representation previously computed ($\rho(P)(\phi) = \mathrm{Ad}_P \circ \phi \circ \mathrm{Ad}_{P^{-1}}$), we define $\bar{\rho} : K \longrightarrow GL(\mathcal{L}^H(\mathfrak{l}_1, \mathfrak{l}_2))$ the following way: for each $Q \in K$, let $\bar{\rho}(Q) = \rho(P)$, where P is any element of $N(H)$ such that $f(P) = Q$. For example, one can take:

$$P = \begin{pmatrix} e^{ix_1} & 0 & 0 & 0 & 0 \\ 0 & e^{i(2x_1-x_3)} & 0 & 0 & 0 \\ 0 & 0 & e^{i(3x_1-x_2)} & 0 & 0 \\ 0 & 0 & 0 & e^{i3x_1} & 0 \\ 0 & 0 & 0 & 0 & e^{-i(9x_1-x_2-x_3)} \end{pmatrix}.$$

We have previously shown that the Higgs field ϕ transforms under the action of $P \in N(H)$ according to:

$$\begin{cases} \phi_{13} \longrightarrow e^{i(-\theta_1+\theta_2+\theta_3-\theta_4)}\phi_{13} \\ \phi_{21} \longrightarrow e^{i(\theta_1-\theta_2-\theta_3+\theta_4)}\phi_{21} \\ \phi_{32} \longrightarrow e^{i(\theta_2-2\theta_3+\theta_4)}\phi_{32}. \end{cases}$$

We deduce that ϕ transforms under the action of $Q \in K$ according to:

$$\begin{cases} \phi_{13} \longrightarrow e^{i(x_1-x_2-x_3)}\phi_{13} \\ \phi_{21} \longrightarrow e^{i(-x_1+x_2+x_3)}\phi_{21} \\ \phi_{32} \longrightarrow e^{i(-x_1+2x_2-x_3)}\phi_{32}. \end{cases}$$

Thus, ϕ_{13}, ϕ_{21} and ϕ_{32} belong respectively to the representations $(1, -1, -1)$, $(-1, 1, 1)$ and $(-1, 2, -1)$ of $\mathbb{T}^3 \simeq N(H)|H$.

The sixth degree term of the potential of the scalar field ϕ is proportional to the squared norm of the tensor $\phi \mathrm{ad}_{12}(\phi)\phi$, which can be identified with the bilinear map $B_\phi : \mathfrak{l}_1 \times \mathfrak{l}_1 \longrightarrow \mathfrak{l}_2$ defined by: $B_\phi(X, X') = \phi(\pi_1[\phi(X), \phi(X')])$.

Let

$$X = \begin{pmatrix} 0 & -\bar{z}_{21} & z_{13} & 0 & 0 \\ z_{21} & 0 & -\bar{z}_{32} & 0 & 0 \\ -\bar{z}_{13} & z_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} 0 & -\bar{z}'_{21} & z'_{13} & 0 & 0 \\ z'_{21} & 0 & -\bar{z}'_{32} & 0 & 0 \\ -\bar{z}'_{13} & z'_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{l}_1.$$

Then:

$$\pi_1[\phi(X), \phi(X')] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & -\bar{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $\gamma = \phi_{13}z_{13}(\phi_{21}z'_{21} + \phi_{32}z'_{32}) - \phi_{13}z'_{13}(\phi_{21}z_{21} + \phi_{32}z_{32})$.

We deduce that:

$$B_\phi(X, X') = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\phi}_{32}\gamma & 0 \\ 0 & 0 & -\phi_{32}\bar{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$\begin{aligned} \bar{\phi}_{32}\gamma &= \bar{\phi}_{32}\phi_{13}z_{13}(\phi_{21}z'_{21} + \phi_{32}z'_{32}) - \bar{\phi}_{32}\phi_{13}z'_{13}(\phi_{21}z_{21} + \phi_{32}z_{32}) \\ &= \bar{\phi}_{32}\phi_{13}\phi_{21}z_{13}z'_{21} + \bar{\phi}_{32}\phi_{13}\phi_{32}z_{13}z'_{32} - \bar{\phi}_{32}\phi_{13}\phi_{21}z'_{13}z_{21} - \bar{\phi}_{32}\phi_{13}\phi_{32}z'_{13}z_{32} \\ &= \bar{\phi}_{32}\phi_{13}\phi_{21}(z_{13}z'_{21} - z'_{13}z_{21}) + \bar{\phi}_{32}\phi_{13}\phi_{32}(z_{13}z'_{32} - z'_{13}z_{32}). \end{aligned}$$

Thus, B_ϕ can be identified with the antisymmetric complex bilinear form (also denoted by B_ϕ):

$$B_\phi : \mathfrak{l}_1 \times \mathfrak{l}_1 \longrightarrow \mathbb{C}$$

defined by: $B_\phi(X, X') = \bar{\phi}_{32}\phi_{13}\phi_{21}(z_{13}z'_{21} - z'_{13}z_{21}) + \bar{\phi}_{32}\phi_{13}\phi_{32}(z_{13}z'_{32} - z'_{13}z_{32})$. $\mathfrak{l}_1$ is isomorphic to the three-dimensional complex vector space $\mathbb{C}^3$. If we set $(e_1, e_2, e_3) = (a_{21}, a_{32}, a_{13})$, then $(e_i)_{1\le i\le 3}$ is a basis of $\mathfrak{l}_1$, and B_ϕ is completely determined by the 3 complex numbers $B_\phi(e_i, e_j)$, $1 \le i < j \le 3$. It is easy to check that:

$$\begin{aligned} &B_\phi(a_{21}, a_{32}) = 0 \\ &B_\phi(a_{21}, a_{13}) = \bar{\phi}_{32}\phi_{13}\phi_{21} \\ &B_\phi(a_{32}, a_{13}) = \bar{\phi}_{32}\phi_{13}\phi_{32}. \end{aligned}$$

Finally, we need to define the $\mathrm{Ad}(N(H))$-invariant scalar products h and k on $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, and use them to compute the squared norm of the tensor $\phi\mathrm{ad}_{12}(\phi)\phi$. We consider the natural $\mathrm{Ad}(SU(5))$-invariant scalar product on $\mathfrak{su}(5)$ defined by: $\langle A, B\rangle = -\frac{1}{2}\mathrm{tr}(AB)$ for every $A, B \in \mathfrak{su}(5)$. Then, h and k are defined to be the restrictions of $\langle , \rangle$ to $\mathfrak{l}_1 \times \mathfrak{l}_1$ and $\mathfrak{l}_2 \times \mathfrak{l}_2$ respectively. Then we take the restriction of k to $\mathfrak{l}_{43}$ and obtain the canonical scalar product on $\mathbb{C} \simeq \mathfrak{l}_{43}$.

Thus, we have in this model: $V^6(\phi) \propto |\bar{\phi}_{32}\phi_{13}\phi_{21}|^2 + |\bar{\phi}_{32}\phi_{13}\phi_{32}|^2$.

5. Conclusion

In the context of Kaluza–Klein theory with, as internal spaces, copies of a homogeneous space G/H, we defined a new type of scalar field as intertwining operators between two $\mathrm{Ad}(N(H))$-invariant complementary subspaces of the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ for a fixed bi-invariant metric on G.

We studied the potential of those scalar fields and found it to be a polynomial of the sixth degree at most, bounded from below when it was exactly of degree 6 or 4. And we found the necessary and sufficient conditions in which the degree is 6 or 4.

We investigated an eventual coupling of the scalar fields with the Yang–Mills strength, and found that there is no such direct coupling if we choose the two complementary subspaces in such a way that all the trivial representation is contained in only one of them. We also computed the kinetic term of the scalar fields and found a natural result. An example leading to an abelian gauge theory was given.

Our scalar fields exhibit therefore properties that are very close to those of the physical Higgs fields, and they have been constructed in a natural way, by looking at hyperbolic directions in the space of Thiry scalar fields, this last one appearing whenever one takes a gravitational theory with some symmetry.

It would seem interesting to carry further the study of the potential of such “canonical Higgs fields”, looking for example at the corresponding symmetry breaking schemes, and seeking what happens at the quantum level. Soliton-like solutions may also appear in the BPS limit, and the introduction of supersymmetry could also be relevant to make contact with certain types of dualities.

This work is being extended to the supersymmetric level, where we can expect to find a more natural origin of our canonical Higgs fields (free from an arbitrary choice of the splitting), arising from supergravity theories.

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Appendix. Example with $G = SU(n)$, $H \simeq U(1)$ and $N(H)|H \simeq \mathbb{T}^{n-2}$

We begin by setting some notation that we are going to use in the following.

- For each $n \in \mathbb{N}^*$, $\mathbb{T}^n$ denotes the n-dimensional torus $U(1) \times \cdots \times U(1)$.
- For each $p \in \mathbb{Z}$, $\mathbb{C}_p$ denotes the vector space $\mathbb{C}$ endowed with the irreducible representation of $U(1)$ labeled by the integer p.
- For $1 \leq j, k \leq n$, E_{jk} denotes the $n \times n$ matrix whose all entries are zero except the (j, k)-entry which is equal to 1.

$U(1)$ is embedded in $SU(n)$ by the following injective homomorphism of Lie groups: $\lambda_{m,l} : U(1) \longrightarrow SU(n)$ defined by:

$$\lambda_{m,l}(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}m\theta}, \mathrm{e}^{\mathrm{i}(m-l)\theta}, \mathrm{e}^{\mathrm{i}(m-2l)\theta}, \ldots, \mathrm{e}^{\mathrm{i}(m-(n-2)l)\theta}, \mathrm{e}^{-\mathrm{i}(n-1)[m-(n-2)\frac{l}{2}]\theta})$$

where m and l are integers. We set $H = \lambda_{m,l}(U(1))$.

We shall focus on the interesting case $l \neq 0$. Here,

$$N(H) = \{\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{n-1}}, \mathrm{e}^{-\mathrm{i}(\theta_1+\cdots+\theta_{n-1})}); \theta_1, \ldots, \theta_{n-1} \in \mathbb{R}\}.$$

Therefore, $N(H) \simeq \mathbb{T}^{n-1}$.

First, we study the adjoint action of H on $\mathfrak{g} = \mathfrak{su}(n)$. Let $h = \lambda_{m,l}(\mathrm{e}^{\mathrm{i}\theta}) \in H$, and $A = [z_{jk}] \in \mathfrak{su}(n)$. We have to compute $hAh^{-1} = [z^h_{jk}]$, which is facilitated by the fact that h is a diagonal matrix:

- For $1 \leq j \leq n-1$, $z^h_{jj} = \mathrm{e}^{\mathrm{i}(m-(j-1)l)\theta} z_{jj} \mathrm{e}^{-\mathrm{i}(m-(j-1)l)\theta}$. Therefore $z^h_{jj} = z_{jj}$ (which is true also for $j = n$), so the diagonal elements of A are not changed.
- For $1 \leq j < k \leq n-1$, $z^h_{jk} = \mathrm{e}^{\mathrm{i}(m-(j-1)l)\theta} z_{jk} \mathrm{e}^{-\mathrm{i}(m-(k-1)l)\theta} = \mathrm{e}^{\mathrm{i}(k-j)l\theta} z_{jk}$.
- For $1 \leq j \leq n-1$ and $k = n$, $z^h_{jn} = \mathrm{e}^{\mathrm{i}(m-(j-1)l)\theta} z_{jn} \mathrm{e}^{-\mathrm{i}(n-1)[m-(n-2)\frac{l}{2}]\theta} = \mathrm{e}^{\mathrm{i}[nm-(n^2-3n+2j)\frac{l}{2}]\theta} z_{jn}$.

We deduce the decomposition of $\mathfrak{g} = \mathfrak{su}(n)$ under the adjoint action of H:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \bigoplus_{r=1}^{n-2} \bigoplus_{j=1}^{n-(r+1)} \mathfrak{l}_{j,j+r} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}$$

where:

$$\begin{aligned} \mathfrak{h} &= \left\{\mathrm{diag}\left(\mathrm{i}m\theta, \mathrm{i}(m-l)\theta, \mathrm{i}(m-2l)\theta, \ldots, \mathrm{i}(m-(n-2)l)\theta, -\mathrm{i}(n-1)\left[m-(n-2)\frac{l}{2}\right]\theta\right); \theta \in \mathbb{R}\right\} \\ &\simeq \mathrm{Lie}(H) \simeq \mathfrak{u}(1) \simeq \mathrm{i}\mathbb{R} \end{aligned}$$

$$\mathfrak{k} = \{\mathrm{diag}(\mathrm{i}x_1, \ldots, \mathrm{i}x_{n-2}, -\mathrm{i}(x_1 + \cdots + x_{n-2}), 0); x_1, \ldots, x_{n-2} \in \mathbb{R}\} \simeq \mathbb{R}^{n-2}$$

$$\mathfrak{l}_{jk} = \{zE_{jk} - \bar{z}E_{kj}; z \in \mathbb{C}\} \simeq \mathbb{C} \quad (j < k)$$

(and $\mathfrak{m} = \mathfrak{k} \oplus \bigoplus_{r=1}^{n-2} \bigoplus_{j=1}^{n-(r+1)} \mathfrak{l}_{j,j+r} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}$)

$\mathfrak{h}$ and $\mathfrak{k}$ carry the trivial representation of H, and:

- for $2 \leq r+1 \leq n-1$ and $1 \leq j \leq n-(r+1)$, $\mathfrak{l}_{j,j+r} \simeq \mathbb{C}_{rl}$
- for $1 \leq s \leq n-1$, $\mathfrak{l}_{sn} \simeq \mathbb{C}_{nm-(n^2-3n+2s)\frac{l}{2}}$.

Second, we study the adjoint action of $N(H)$ on $\mathfrak{g} = \mathfrak{su}(n)$. Noticing that $N(H) \simeq \mathbb{T}^{n-1}$ is nothing but the natural maximal torus of $\mathfrak{su}(n)$, we see immediately that the decomposition of $\mathfrak{g}$ under $N(H)$ reads:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{k} \oplus \bigoplus_{1 \le j < k \le n-1} \mathfrak{l}_{jk} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}$$

where:

- $\mathfrak{l}_{jk} \simeq \mathbb{C}$ carrying the irreducible representation of $\mathbb{T}^{n-1}$ labeled by the element $(0, \dots, 1, \dots, -1, \dots, 0) \in \mathbb{Z}^{n-1}$, 1 being in the jth place, -1 in the kth place, and 0 in the $n-3$ remaining places.
- $\mathfrak{l}_{sn} \simeq \mathbb{C}$ carrying the irreducible representation of $\mathbb{T}^{n-1}$ labeled by the element $(1, \dots, 2, \dots, 1, \dots, 1) \in \mathbb{Z}^{n-1}$, 2 being in the sth place, and 1 in the $n-2$ remaining places.

Let us define the two following subspaces of $\mathfrak{g}$:

$$\mathfrak{m}_1 = \mathfrak{k} \oplus \bigoplus_{r=1}^{n-3} \bigoplus_{j=1}^{n-(r+2)} \mathfrak{l}_{j,j+r}$$

$$\mathfrak{m}_2 = \mathfrak{l}_{1,n-1} \oplus \bigoplus_{r=1}^{n-3} \mathfrak{l}_{n-1-r,n-1} \oplus \bigoplus_{s=1}^{n-1} \mathfrak{l}_{sn}.$$

It is clear that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are $\mathrm{Ad}(N(H))$-invariant complementary subspaces of $\mathfrak{m}$, and thus we have an $\mathrm{Ad}(N(H))$-invariant splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. We deduce a space of Higgs fields:

$$\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = \bigoplus_{r=1}^{n-3} \mathcal{L}\left(\bigoplus_{j=1}^{n-(r+2)} \mathfrak{l}_{j,j+r}, \mathfrak{l}_{n-1-r,n-1}\right).$$

The dimension of $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$ can be easily computed:

$$\dim \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2) = 2\sum_{r=1}^{n-3}[n-(r+2)].1 = (n-2)(n-3).$$

A natural basis for the vector space $\mathfrak{m}$ is given by the following matrices:

$$\mathrm{i}(E_{jj} - E_{n-1,n-1}), \quad 1 \le j \le n-2 \text{ (basis for } \mathfrak{k}\text{)}.$$
$$a_{jk} = E_{jk} - E_{kj} \quad \text{and} \quad b_{jk} = \mathrm{i}(E_{jk} + E_{kj}), \quad 1 \le j < k \le n \text{ (basis for } \mathfrak{l}_{jk}\text{)}.$$

This basis is adapted to our decompositions, since the vectors:

$\mathrm{i}(E_{jj} - E_{n-1,n-1})$, $1 \le j \le n-2$, $a_{j,j+r}$ and $b_{j,j+r}$ for $1 \le r \le n-3$ and $1 \le j \le n-(r+2)$ form a basis of the subspace $\mathfrak{m}_1$, and the vectors:

$a_{1,n-1}, b_{1,n-1}, a_{n-1-r,n-1}$ and $b_{n-1-r,n-1}$ for $1 \le r \le n-3$, a_{sn} and b_{sn} for $1 \le s \le n-1$ form a basis of the subspace $\mathfrak{m}_2$.

Let $\phi \in \mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. We have $\phi(i(E_{jj} - E_{n-1,n-1})) = 0$.

We define the real numbers $\varphi_{j,j+r}, \psi_{j,j+r}$ for $1 \le r \le n-3$ and $1 \le j \le n-(r+2)$ by:

$$\phi(a_{j,j+r}) = \varphi_{j,j+r} a_{n-1-r,n-1} + \psi_{j,j+r} b_{n-1-r,n-1}$$
$$\phi(b_{j,j+r}) = -\psi_{j,j+r} a_{n-1-r,n-1} + \varphi_{j,j+r} b_{n-1-r,n-1}.$$

The $\varphi_{j,j+r}$ and $\psi_{j,j+r}$ constitute the $(n-2)(n-3)$ components of the Higgs field ϕ in the bases of $\mathfrak{m}_1$ and $\mathfrak{m}_2$ defined above.

Third, we study the representation of $N(H)|H$ on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$. For this, we first look at the representation of $N(H)$ on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$:

Given $P = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_{n-1}}, \mathrm{e}^{-\mathrm{i}(\theta_1 + \dots + \theta_{n-1})}) \in N(H)$, and $A = [z_{jk}] \in \mathfrak{m}_1$, we begin by computing $\mathrm{Ad}_{P^{-1}}(A) = P^{-1}AP = [z_{jk}^{P^{-1}}]$, which again is facilitated because P is diagonal.

- For $1 \le j \le n-1$, $z_{jj}^{P^{-1}} = \mathrm{e}^{-\mathrm{i}\theta_j} z_{jj} \mathrm{e}^{\mathrm{i}\theta_j}$. Therefore $z_{jj}^{P^{-1}} = z_{jj}$ (which is true also for $j = n$), so the diagonal elements of A are not changed.
- For $1 \le j < k \le n-2$, $z_{jk}^{P^{-1}} = \mathrm{e}^{-\mathrm{i}\theta_j} z_{jk} \mathrm{e}^{\mathrm{i}\theta_k} = \mathrm{e}^{\mathrm{i}(-\theta_j + \theta_k)} z_{jk}$.

Then, we compute $\mathrm{Ad}_P \circ \phi \circ \mathrm{Ad}_{P^{-1}}(A)$. Writing

$$A = \sum_{j=1}^{n-2} x_j i(E_{jj} - E_{n-1,n-1}) + \sum_{r=1}^{n-3} \sum_{j=1}^{n-(r+2)} (x_{j,j+r} a_{j,j+r} + y_{j,j+r} b_{j,j+r})$$

and noting that $x_{jk} a_{jk} + y_{jk} b_{jk} = z_{jk} E_{jk} - \bar{z}_{jk} E_{kj}$ if $z_{jk} = x_{jk} + \mathrm{i} y_{jk}$, we see that:

$$\mathrm{Ad}_{P^{-1}}(A) = \sum_{j=1}^{n-2} x_j \mathrm{i}(E_{jj} - E_{n-1,n-1}) + \sum_{r=1}^{n-3} \sum_{j=1}^{n-(r+2)} (\mathrm{e}^{\mathrm{i}(-\theta_j + \theta_{j+r})} z_{j,j+r} E_{j,j+r} - \mathrm{e}^{-\mathrm{i}(-\theta_j + \theta_{j+r})} \bar{z}_{j,j+r} E_{j+r,j})$$

and we deduce, letting $\phi_{jk} = \varphi_{jk} + \mathrm{i}\psi_{jk}$, that:

$$\phi(\mathrm{Ad}_{P^{-1}}(A)) = \sum_{r=1}^{n-3} \left(\sum_{j=1}^{n-(r+2)} \mathrm{e}^{\mathrm{i}(-\theta_j + \theta_{j+r})} \phi_{j,j+r} z_{j,j+r} E_{n-1-r,n-1} - \sum_{j=1}^{n-(r+2)} \mathrm{e}^{-\mathrm{i}(-\theta_j + \theta_{j+r})} \bar{\phi}_{j,j+r} \bar{z}_{j,j+r} E_{n-1,n-1-r} \right).$$

Finally,

$$\mathrm{Ad}_P(\phi(\mathrm{Ad}_{P^{-1}}(A))) = \sum_{r=1}^{n-3} \left(\sum_{j=1}^{n-(r+2)} \mathrm{e}^{\mathrm{i}(\theta_{n-1-r,n-1} - \theta_{n-1} - \theta_j + \theta_{j+r})} \phi_{j,j+r} z_{j,j+r} E_{n-1-r,n-1} - \sum_{j=1}^{n-(r+2)} \mathrm{e}^{-\mathrm{i}(\theta_{n-1-r,n-1} - \theta_{n-1} - \theta_j + \theta_{j+r})} \bar{\phi}_{j,j+r} \bar{z}_{j,j+r} E_{n-1,n-1-r} \right).$$

We define the homomorphism of Lie groups $f : N(H) \longrightarrow G$ by:

$$f(P) = \mathrm{diag}(\mathrm{e}^{\mathrm{i}[(1-(n-2)\frac{l}{m})\theta_1 - \theta_{n-1}]}, \ldots, \mathrm{e}^{\mathrm{i}[(1-\frac{l}{m})\theta_1 - \theta_2]}, \mathrm{e}^{-\mathrm{i}[(n-2)(1-(n-1)\frac{l}{2m})\theta_1 - (\theta_2 + \cdots + \theta_{n-1})]}, 1)$$

for each $P = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{n-1}}, \mathrm{e}^{-\mathrm{i}(\theta_1 + \cdots + \theta_{n-1})}) \in N(H)$.

We see that $\mathrm{Ker} f = H$, and if we set $K = \{\mathrm{diag}(\mathrm{e}^{\mathrm{i}x_1}, \ldots, \mathrm{e}^{\mathrm{i}x_{n-2}}, \mathrm{e}^{-\mathrm{i}(x_1+\cdots+x_{n-2})}, 1); x_1, \ldots, x_{n-2} \in \mathbb{R}\}$, then $K = \mathrm{Im} f$, and we obtain a surjective homomorphism that we also denote by f. Thus, $f : N(H) \longrightarrow K$ induces an isomorphism $\bar{f} : N(H)|H \longrightarrow K$. It is easy now to explicit the action of $N(H)|H$ on $\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2)$: if $\rho : N(H) \longrightarrow GL(\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2))$ is the representation previously computed ($\rho(P)(\phi) = \mathrm{Ad}_P \circ \phi \circ \mathrm{Ad}_{P^{-1}}$), we define $\bar{\rho} : K \longrightarrow GL(\mathcal{L}^H(\mathfrak{m}_1, \mathfrak{m}_2))$ the following way: for each $Q = \mathrm{diag}(\mathrm{e}^{\mathrm{i}x_1}, \ldots, \mathrm{e}^{\mathrm{i}x_{n-2}}, \mathrm{e}^{-\mathrm{i}(x_1+\cdots+x_{n-2})}, 1) \in K$, let $\bar{\rho}(Q) = \rho(P)$, where P is any element of $N(H)$ such that $f(P) = Q$. For example, one can take $P = \mathrm{diag}(\mathrm{e}^{\mathrm{i}x_1}, \mathrm{e}^{\mathrm{i}[(1-\frac{l}{m})x_1 - x_{n-2}]}, \ldots, \mathrm{e}^{\mathrm{i}[(1-(n-3)\frac{l}{m})x_1 - x_2]}, \mathrm{e}^{-\mathrm{i}(n-2)\frac{l}{m}x_1}, \mathrm{e}^{-\mathrm{i}[(n-2)(1-(n-1)\frac{l}{2m})x_1 - (x_2+\cdots+x_{n-2})]})$.

We have shown that the Higgs field ϕ transform under the action of $P = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{n-1}}, \mathrm{e}^{-\mathrm{i}(\theta_1+\cdots+\theta_{n-1})}) \in N(H)$ according to:

$$\phi_{j,j+r} \longrightarrow \mathrm{e}^{\mathrm{i}(\theta_{n-1-r,n-1} - \theta_{n-1} - \theta_j + \theta_{j+r})} \phi_{j,j+r}.$$

Using the relations:

$$\begin{cases} \theta_j = \left(1 - (j-1)\dfrac{l}{m}\right) x_1 - x_{n-j} \quad (\text{for } j \geq 2) \\ \theta_{j+r} = \left(1 - (j+r-1)\dfrac{l}{m}\right) x_1 - x_{n-j-r} \\ \cdots \\ \theta_{n-1-r} = \left(1 - (n-r-2)\dfrac{l}{m}\right) x_1 - x_{r+1} \\ \theta_{n-1} = \left(1 - (n-2)\dfrac{l}{m}\right) x_1 - x_1 \quad \left(= -(n-2)\dfrac{l}{m} x_1\right). \end{cases}$$

We derive the transformation rules of ϕ under the action of $Q = \mathrm{diag}(e^{ix_1}, \dots, e^{ix_{n-2}}, e^{-i(x_1+\cdots+x_{n-2})}, 1)$:

$$\phi_{j,j+r} \longrightarrow e^{i(x_1-x_{r+1}+x_{n-j}-x_{n-j-r})}\phi_{j,j+r} \quad (\text{for } j \geq 2).$$

Our final task is the computation of the Higgs potential of this model. For this, we shall need first some structure constants which are given by the following lemma:

Lemma A.1. (0) $[E_{jk}, E_{j'k'}] = \delta_{kj'}E_{jk'} - \delta_{k'j}E_{j'k}$
(1) $[a_{jk}, a_{j'k'}] = \delta_{kj'}a_{jk'} - \delta_{k'j}a_{j'k} - \delta_{kk'}a_{jj'} - \delta_{jj'}a_{kk'}$
(2) $[a_{jk}, b_{j'k'}] = \delta_{kj'}b_{jk'} - \delta_{k'j}b_{j'k} + \delta_{kk'}b_{jj'} - \delta_{jj'}b_{kk'}$
(3) $[b_{jk}, a_{j'k'}] = \delta_{kj'}b_{jk'} - \delta_{k'j}b_{j'k} - \delta_{kk'}b_{jj'} + \delta_{jj'}b_{kk'}$
(4) $[b_{jk}, b_{j'k'}] = -\delta_{kj'}a_{jk'} + \delta_{k'j}a_{j'k} - \delta_{kk'}a_{jj'} - \delta_{jj'}a_{kk'}$.

Proof. (0) follows immediately by expressing the matrix product of E_{jk} and $E_{j'k'}$ in terms of Kronecker symbols. For (1),

$$\begin{aligned}
[a_{jk}, a_{j'k'}] &= [E_{jk} - E_{kj}, E_{j'k'} - E_{k'j'}] \\
&= [E_{jk}, E_{j'k'}] - [E_{jk}, E_{k'j'}] - [E_{kj}, E_{j'k'}] + [E_{kj}, E_{k'j'}] \\
&= \delta_{kj'}E_{jk'} - \delta_{k'j}E_{j'k} - \delta_{kk'}E_{jj'} + \delta_{j'j}E_{k'k} - \delta_{jj'}E_{kk'} + \delta_{k'k}E_{j'j} + \delta_{jk'}E_{kj'} - \delta_{j'k}E_{k'j} \\
&= \delta_{kj'}a_{jk'} - \delta_{k'j}a_{j'k} - \delta_{kk'}a_{jj'} - \delta_{jj'}a_{kk'}.
\end{aligned}$$

The computations are similar for (2), (3) and (4). $\square$

Next, we have to find the components of the tensor $\phi\mathrm{ad}_{12}(\phi)\phi$, which can be viewed as the bilinear map $B_\phi : \mathfrak{m}_1 \times \mathfrak{m}_1 \longrightarrow \mathfrak{m}_2$ defined by: $B_\phi(X, Y) = \phi(\pi_1[\phi(X), \phi(Y)])$. Thus, we need to express the components of $B_\phi(a_{j,j+r}, a_{j',j'+r'})$, $B_\phi(a_{j,j+r}, b_{j',j'+r'})$, etc. in the corresponding basis of $\mathfrak{m}_2$

$$\begin{aligned}
&[\phi(a_{j,j+r}), \phi(a_{j',j'+r'})] \\
&\quad= [\varphi_{j,j+r}a_{n-1-r,n-1} + \psi_{j,j+r}b_{n-1-r,n-1}, \varphi_{j',j'+r'}a_{n-1-r',n-1} + \psi_{j',j'+r'}b_{n-1-r',n-1}] \\
&\quad= \varphi_{j,j+r}\varphi_{j',j'+r'}(\delta_{r'0}a_{n-1-r,n-1} - \delta_{r0}a_{n-1-r',n-1} - a_{n-1-r,n-1-r'} - \delta_{rr'}a_{n-1,n-1}) \\
&\qquad+ \varphi_{j,j+r}\psi_{j',j'+r'}(\delta_{r'0}b_{n-1-r,n-1} - \delta_{r0}b_{n-1-r',n-1} + b_{n-1-r,n-1-r'} - \delta_{rr'}b_{n-1,n-1}) \\
&\qquad+ \psi_{j,j+r}\varphi_{j',j'+r'}(\delta_{r'0}b_{n-1-r,n-1} - \delta_{r0}b_{n-1-r',n-1} - b_{n-1-r,n-1-r'} + \delta_{rr'}b_{n-1,n-1}) \\
&\qquad+ \psi_{j,j+r}\psi_{j',j'+r'}(-\delta_{r'0}a_{n-1-r,n-1} + \delta_{r0}a_{n-1-r',n-1} - a_{n-1-r,n-1-r'} - \delta_{rr'}a_{n-1,n-1}).
\end{aligned}$$

In particular, noticing that $a_{jk} = -a_{kj}$ (which implies $a_{jj} = 0$) and that $b_{jk} = b_{kj}$, we have:

$$\begin{aligned}
&[\phi(a_{j,j+r}), \phi(a_{j',j'})] = 2\varphi_{j,j+r}\psi_{j',j'}b_{n-1-r,n-1} - 2\psi_{j,j+r}\psi_{j',j'}a_{n-1-r,n-1} \quad \text{if } r \neq 0 \\
&[\phi(a_{j,j}), \phi(a_{j',j'+r'})] = -2\psi_{j,j}\varphi_{j',j'+r'}b_{n-1-r',n-1} + 2\psi_{j,j}\psi_{j',j'+r'}a_{n-1-r',n-1} \quad \text{if } r' \neq 0 \\
&[\phi(a_{j,j}), \phi(a_{j',j'})] = 0 \\
&[\phi(a_{j,j+r}), \phi(a_{j',j'+r})] = (\varphi_{j,j+r}\psi_{j',j'+r'} - \psi_{j,j+r}\varphi_{j',j'+r})(b_{n-1-r,n-1-r} - b_{n-1,n-1}) \quad \text{if } r \neq 0 \\
&[\phi(a_{j,j+r}), \phi(a_{j',j'+r'})] = -\varphi_{j,j+r}\varphi_{j',j'+r'}a_{n-1-r,n-1-r'} + \varphi_{j,j+r}\psi_{j',j'+r'}b_{n-1-r,n-1-r'} \\
&\qquad - \psi_{j,j+r}\varphi_{j',j'+r'}b_{n-1-r,n-1-r'} - \psi_{j,j+r}\psi_{j',j'+r'}a_{n-1-r,n-1-r'} \\
&\qquad \text{if } r \neq 0,\ r' \neq 0 \text{ and } r \neq r'.
\end{aligned}$$

If $1 \leq r - r' \leq n - 3$ and $1 \leq n - 1 - r \leq n - (r - r' + 2)$ then $a_{n-1-r,n-1-r'} \in \mathfrak{m}_1$ and $b_{n-1-r,n-1-r'} \in \mathfrak{m}_1$. Therefore,

$$\begin{aligned}
\pi_1([\phi(a_{j,j+r}), \phi(a_{j',j'+r'})]) &= (-\varphi_{j,j+r}\varphi_{j',j'+r'} - \psi_{j,j+r}\psi_{j',j'+r'})a_{n-1-r,n-1-r'} \\
&\quad+ (\varphi_{j,j+r}\psi_{j',j'+r'} - \psi_{j,j+r}\varphi_{j',j'+r'})b_{n-1-r,n-1-r'}.
\end{aligned}$$

Using the fact that:

$$\begin{aligned}
\phi(a_{n-1-r,n-1-r'}) &= \varphi_{n-1-r,n-1-r'}a_{n-1-(r-r'),n-1} + \psi_{n-1-r,n-1-r'}b_{n-1-(r-r'),n-1} \\
\phi(b_{n-1-r,n-1-r'}) &= -\psi_{n-1-r,n-1-r'}a_{n-1-(r-r'),n-1} + \varphi_{n-1-r,n-1-r'}b_{n-1-(r-r'),n-1}
\end{aligned}$$

we deduce:

$$\begin{aligned}B_\phi(a_{j,j+r},a_{j',j'+r'}) = {} & (-\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r}-\psi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\ & -\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}+\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'})a_{n-1-(r-r'),n-1}\\ & +(-\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\ & +\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'})b_{n-1-(r-r'),n-1}.\end{aligned}$$

Completely analogous calculations lead to:

$$\begin{aligned}B_\phi(a_{j,j+r},b_{j',j'+r'}) = {} & (\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\ & -\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'})a_{n-1-(r-r'),n-1}\\ & +(\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\ & +\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}+\psi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'})b_{n-1-(r-r'),n-1}\\ B_\phi(b_{j,j+r},a_{j',j'+r'}) = {} & (\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\ & +\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}+\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'})a_{n-1-(r-r'),n-1}\\ & +(\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\ & -\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'})b_{n-1-(r-r'),n-1}\\ B_\phi(b_{j,j+r},b_{j',j'+r'}) = {} & (\psi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\ & +\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'})a_{n-1-(r-r'),n-1}\\ & +(-\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\ & -\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}+\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'})b_{n-1-(r-r'),n-1}.\end{aligned}$$

Finally, we need to define the $\mathrm{Ad}(N(H))$-invariant scalar products h and k on $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively, and use them to compute the squared norm of the tensor $\phi\mathrm{ad}_{12}(\phi)\phi$. We consider the natural $\mathrm{Ad}(SU(n))$-invariant scalar product on $\mathfrak{su}(n)$ defined by: $\langle A,B\rangle = -\frac{1}{2}\mathrm{tr}(AB)$ for every $A,B\in\mathfrak{su}(n)$. Then, h and k are defined to be the restrictions of $\langle,\rangle$ to $\mathfrak{m}_1\times\mathfrak{m}_1$ and $\mathfrak{m}_2\times\mathfrak{m}_2$ respectively.

Lemma A.2. *The a_{jk} and b_{jk} constitute an orthonormal family in $\mathfrak{su}(n)$, i.e.*

(1) $\langle a_{jk},a_{j'k'}\rangle = \delta_{jj'}\delta_{kk'}$
(2) $\langle a_{jk},b_{j'k'}\rangle = 0$
(3) $\langle b_{jk},a_{j'k'}\rangle = 0$
(4) $\langle b_{jk},b_{j'k'}\rangle = \delta_{jj'}\delta_{kk'}$.

Proof. First, notice that $\mathrm{tr}(E_{jk}) = \delta_{jk}$ and that the a_{jk}, b_{jk} are only defined for $j<k$. We prove only (1), the calculations being similar for (2), (3) and (4)

$$\begin{aligned}a_{jk}a_{j'k'} &= (E_{jk}-E_{kj})(E_{j'k'}-E_{k'j'})\\ &= E_{jk}E_{j'k'}-E_{jk}E_{k'j'}-E_{kj}E_{j'k'}+E_{kj}E_{k'j'}\\ &= \delta_{kj'}E_{jk'}-\delta_{kk'}E_{jj'}-\delta_{jj'}E_{kk'}+\delta_{jk'}E_{kj'}\\ \langle a_{jk},a_{j'k'}\rangle &= -\frac{1}{2}\mathrm{tr}(a_{jk}a_{j'k'})\\ &= -\frac{1}{2}(\delta_{kj'}\delta_{jk'}-\delta_{kk'}\delta_{jj'}-\delta_{jj'}\delta_{kk'}+\delta_{jk'}\delta_{kj'})\\ &= \delta_{jj'}\delta_{kk'}.\quad\square\end{aligned}$$

h and k are thus represented by the identity matrices in the corresponding bases of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, and the sixth-degree term of the Higgs potential $V^{(6)}(\phi) = \frac{1}{4}\|\phi\mathrm{ad}_{12}(\phi)\phi\|^2_{hhk}$ reads in this model:

$$\begin{aligned}V^{(6)}(\phi) = \frac{1}{4}\sum\{ & (-\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r}-\psi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\ & -\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}+\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'})^2\\ & +(-\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\ & +\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'})^2\end{aligned}$$

$$
\begin{aligned}
&\times(\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\
&-\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'})^2\\
&+(\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\
&+\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}+\psi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'})^2\\
&\times(\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\
&+\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}+\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'})^2\\
&+(\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\
&-\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'})^2\\
&\times(\psi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}\\
&+\psi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'})^2\\
&+(-\psi_{j,j+r}\psi_{j',j'+r'}\psi_{n-1-r,n-1-r'}-\varphi_{j,j+r}\varphi_{j',j'+r'}\psi_{n-1-r,n-1-r'}\\
&-\psi_{j,j+r}\varphi_{j',j'+r'}\varphi_{n-1-r,n-1-r'}+\varphi_{j,j+r}\psi_{j',j'+r'}\varphi_{n-1-r,n-1-r'})^2\}
\end{aligned}
$$

the sum being made on the positive integers j, r, j', r' satisfying the following constraints:

$$
\left\{\begin{matrix}
1\le r\le n-3\\
1\le j\le n-(r+2)\\
1\le r'\le n-3\\
1\le j'\le n-(r'+2)\\
1\le r-r'\le n-3\\
1\le n-1-r\le n-(r-r'+2)
\end{matrix}\right\}.
$$

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